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Explicit Solutions of Galois Embedding Problems by Means of Generalized Clifford Algebras

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In this paper we study Galois embedding problems given by central extensions with cyclic kernel. We find a new expression for the obstruction to the solvability of these embedding problems in terms of Galois symbols. We also give a method to construct the solutions when these problems are solvable. We find a solution from a coordinate of the norm of an adequate element in a generalized Clifford algebra.

1. Introduction

Let $n \geq 2$ be an integer and K be a field of characteristic not dividing n containing the group μ_n of nth roots of unity.

Let L/K be a Galois extension of finite degree with $\Gamma = \mathrm{Gal}(L/K)$. Let

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \Gamma \to 0$$

be a central extension of Γ. Let $j : G_K \to \Gamma$ be the surjective homomorphism corresponding to L/K. We consider the homomorphism between the cohomology groups $j_2^* : H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to H^2(G_K, \mathbb{Z}/n\mathbb{Z})$ induced by j and let ε be the element in $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ corresponding to (E).

We consider the embedding problem given by $L/K, \Gamma, (E)$. It is solvable if and only if $j_2^*(\varepsilon) = 0$: see Malle and Matzat (1999). The element $j_2^*(\varepsilon)$ is called the obstruction to the solvability of the embedding problem. In general, the element $j_2^*(\varepsilon)$ is hard to compute. A formula of Serre (1984) expresses the obstruction to the solvability of embedding problems given by the double cover of the alternating and symmetric groups with kernel $\mathbb{Z}/2\mathbb{Z}$ in terms of Hasse–Witt invariants. Fröhlich (1985) generalizes Serre's result to embedding problems (with kernel $\mathbb{Z}/2\mathbb{Z}$) associated to orthogonal representations of Galois groups. Fröhlich's result has been applied by Crespo (1990b) and by Swallow (1995). A study for embedding problems with kernel $\mathbb{Z}/p\mathbb{Z}$ for p a prime integer was done by Massy (1985). Swallow (1996b) generalizes Massy's paper using projective representations and Galois cohomology. Crespo (1989, 1990a) constructs explicitly the solutions to solvable embedding problems with kernel $\mathbb{Z}/2\mathbb{Z}$ of the type studied by Serre and in Crespo (1990b) generalizes her method to the case of solvable embedding problems associated to orthogonal representations: see also Swallow (1996a).

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In this paper we study Galois embedding problems given by central extensions with cyclic kernel. Our main achievements are:

— A new expression for the obstruction to the solvability of these embedding problems in terms of Galois symbols, which allows its computation.

— A method for constructing the solutions when these problems are solvable generalizing the results obtained by Crespo. To construct the solutions we develop the package `n-Clifford`, a set of programs in *Mathematica* designed to perform explicit calculations.

To the embedding problem considered, we associate a representation in the group of graded automorphisms of an adequate generalized Clifford algebra. We find the expression for the obstruction in terms of Brauer invariants of generalized Clifford algebras. We obtain a solution of the solvable embedding problems of the form $L(\sqrt[n]{\gamma})$ where γ is a coordinate of the norm of an adequate element in a generalized Clifford algebra.

Therefore, it is necessary to introduce the generalized Clifford algebras and to study their structure, invariant and graded automorphisms. This study in general for a central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebra is carried out in Vela (1998a).

In the first sections we study generalized Clifford algebras as central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebras, we define their norm and admissible norms and Brauer invariant.

We then study the representations of a profinite group in the graded automorphism group of a generalized Clifford algebra and define some cohomology classes associated to it. We also define the twisted algebra by a representation and find a formula to relate the Brauer invariants of these algebras and the classes defined.

By finding adequate representations and applying the above formula, we compute the obstruction to the solvability of embedding problems. Finally, we give a method for obtaining explicitly the solutions when these problems are solvable, and present some examples.

We fix ω a primitive nth root of unity. We denote by lg the group homomorphism $\lg : \mu_n \longrightarrow \mathbb{Z}/n\mathbb{Z}$ where $\lg(\omega^i) = i$. The separable closure of K is denoted by K^{sep} and the absolute Galois group is denoted by $G_K = \mathrm{Gal}(K^{sep}/K)$.

2. Graded Automorphisms in Generalized Clifford Algebras

We use the term *generalized Clifford algebra* to refer to a finite-dimensional K-algebra generated by elements $\{e_1, \ldots, e_m\}$ with relations $e_i^n = a_i \in K^*$ and $e_i e_j = \omega e_j e_i$ if $i < j$ (see Vela, 1998a). The elements $e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}, \varepsilon_i \in \{0, \ldots, n-1\}$ form a basis of $C_m^{(n)}(V)$ as K-vector space. We denote this algebra as $C_m^{(n)}(e_1, \ldots, e_m)$, $C_m^{(n)}(a_1, \ldots, a_m)$ or $C_m^{(n)}(V)$ for $V = \langle e_1, \ldots, e_m \rangle_K$. When m or n are fixed we omit them in notation.

We can make $C(V)$ into a $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra $C(V) = C(V)_0 \oplus \cdots \oplus C(V)_{n-1}$, by setting $C(V)_l = \langle e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m} \mid \varepsilon_1 + \cdots + \varepsilon_m \equiv l \pmod{n} \rangle_K$.

If A is a $\mathbb{Z}/n\mathbb{Z}$-graded algebra, the elements in $h(A) = A_0 \cup \cdots \cup A_{n-1}$ are called the *homogeneous elements of* A. In particular, the elements in $h(C(V))$ are the *homogeneous elements of* $C(V)$. If $x \in A_l$, we put $\partial(x) = l$ and say that l is *the degree of* x.

If A, B are two $\mathbb{Z}/n\mathbb{Z}$-graded algebras, *the graded tensor product of* A and B, denoted by $A \hat{\otimes} B$, is a $\mathbb{Z}/n\mathbb{Z}$-graded algebra such that:

(a) The i-direct summand ($i = 0, \ldots, n-1$) is $(A\hat{\otimes}B)_i = \displaystyle\bigoplus_{j+k\equiv i \pmod n} (A_j \otimes B_k)$.

(b) The multiplication is induced by $(a\hat{\otimes}b)(a'\hat{\otimes}b') = \omega^{-\partial(b)\partial(a')}aa'\hat{\otimes}bb'$ (a, b homog.).

For generalized Clifford algebras, there exists a graded algebra isomorphism

$$C(e_1, \ldots, e_m) \simeq C(e_1)\hat{\otimes}\cdots\hat{\otimes}C(e_m).$$

The dimension of $C_m^{(n)}(V)$ is n^m.

The definition of the generalized Clifford algebra depends on the basis of the vector space V. Below we look at this dependence in more detail.

We call a basis $\{v_1, \ldots, v_m\}$ of V an *admissible basis* if in $C(V)$ it holds that $v_i^n \in K^*$ ($i = 1, \ldots, m$) and $v_iv_j = \omega v_jv_i$ for $i < j$. If $n > 2$, the only base changes in V to an admissible basis $\{e_1, \ldots, e_m\}$ from another admissible basis $\{v_1, \ldots, v_m\}$ are of the form $v_i = \alpha_ie_i$ with $\alpha_i \in K$ for $i = 1, \ldots, m$. If $n = 2$, the admissible bases are the orthogonal bases with respect to the quadratic form.

Generalized Clifford algebras are central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebras. We recall that a $\mathbb{Z}/n\mathbb{Z}$-graded algebra A is central simple if it does not have graded proper ideals and the centre $Z(A)$ has no elements of degree 0 other than the elements in K.

If A is a central simple graded algebra of odd type (that is $Z(A) \cap A_1 \neq 0$) we can express $Z(A) = K \oplus \cdots \oplus Kz^{n-1}$ where $z \in Z(A) \cap A_1$ and $z^n = a \in K^*$ (Vela, 1998a, Theorem 1.4.4). Such an element z is called a *structure element of* A and the element a is called the *invariant of* A. If A is a central simple graded algebra of even type (that is $Z(A) = K$) we can express the centre of A_0 as $Z(A_0) = K \oplus \cdots \oplus Kz^{n-1}$ where $z \in Z(A_0)$ and $z^n = a \in K^*$ (Vela, 1998a, Theorem 1.4.5). Such an element z is called a *structure element of* A and the element a is called the *invariant of* A. We can take this structure element as the element z such that $\nu(x) = z^{-1}xz$, for all $x \in A$, where ν is the automorphism of A defined by $\nu(x) = \omega^{\partial(x)}x$ for x an homogeneous element of A.

Generalized Clifford algebras are central simple graded algebras of even type (that is $Z(C(V)) = K$) if m is even and of odd type (that is $Z(C(V)) \cap [C(V)]_1 \neq 0$) if m is odd.

Translating general results of central simple graded algebras (see Vela, 1998a), we obtain structure theorems for generalized Clifford algebras. In particular, we obtain for them a structure element and the invariant as follows.

Theorem 2.1. *Let $C(V)$ be a generalized Clifford algebra.*

(a) If $C(V)$ is of odd type, a structure element is $z = e_1e_2^{-1}e_3\cdots e_{m-1}^{-1}e_m \in C(V)_1$ and its invariant is

$$a = z^n = (-1)^{\frac{(n-1)(m-1)}{2}}a_1a_2^{-1}\cdots a_{m-1}^{-1}a_m.$$

(b) If $C(V)$ is of even type, a structure element is $z = e_1^{-1}e_2\cdots e_{m-1}^{-1}e_m \in C(V)_0$ and its invariant is

$$a = z^n = (-1)^{\frac{(n-1)m}{2}}a_1^{-1}a_2\cdots a_{m-1}^{-1}a_m.$$

We now recall some properties about graded automorphisms of generalized Clifford algebras studied in Vela (1998a). Particular cases have been studied in Vela (1998b). Let $f : C(V) \to C(V)$ be an automorphism. We say that f is a graded automorphism if $f(C(V)_i) \subseteq C(V)_i$ for all $i \in \{0, \ldots, n-1\}$.

Proposition 2.2. *The sequence*

$$1 \to K^* \to \{\textit{homogeneous elements of } C(V)^*\} \xrightarrow{\varphi} \mathrm{Autgr}(C(V)) \to 1$$

is exact, where, for all $x \in C(V)$ *homogeneous*

(a) $\varphi(s)(x) = sxs^{-1}$ *if* $\dim(V)$ *is even,*
(b) $\varphi(s)(x) = \omega^{\partial(s)\partial(x)} sxs^{-1}$ *if* $\dim(V)$ *is odd.*

Proposition 2.3. *Suppose* $n > 2$.

(a) Let $f \in \mathrm{Autgr}(C(V))$ *be such that* $f(V) \subset V$. *Then* $f(e_i) = \alpha_i e_i$ *for all* i, *with* $\alpha_i \in \mu_n$.
(b) If s *is an homogeneous element of* $C(V)^*$ *such that* $\varphi(s)$ *leaves* V *invariant, then* $s = \lambda e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}$ *for some* $\lambda \in K$ *and* $\varepsilon_i \in \{0, \ldots, n-1\}$.

Proof. (a) This part is straightforward.

(b) If the dimension of V is even, $\varphi(s) \equiv s(\cdot)s^{-1}$. Since $\varphi(s)(V) \subset V$, $sVs^{-1} \subset V$ and then $se_i = \alpha_i e_i s$.

Let $s = \displaystyle\sum_{\varepsilon_1+\cdots+\varepsilon_m \equiv k(n)} \lambda_{\varepsilon_1 \cdots \varepsilon_m} e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}$ be homogeneous of degree k, with $0 \le \varepsilon_i < n$.

Since $se_1 = \alpha_1 e_1 s$, we have for every $(\varepsilon_1, \ldots, \varepsilon_m)$, $\alpha_1 = \omega^{-(\varepsilon_2+\cdots+\varepsilon_m)} = \omega^{-(k-\varepsilon_1)}$ and $\omega^{\varepsilon_1} = \omega^k \alpha_1$. Thus, since $\varepsilon_1 < n$, the automorphism $\varphi(s)$, in particular α_1, determines ε_1. In the same way we prove that all ε_j are determined by f_s, because $\omega^{\varepsilon_j} = \alpha_j \omega^k \omega^{-2(\varepsilon_1+\cdots+\varepsilon_{j-1})}$. Therefore, all ε_j are unique, and then s has only one summand.

If the dimension of V is odd, we have $f_s(x) = \omega^{\partial(s)\partial(x)} sxs^{-1}$. Because of $f_s(e_i) = \omega^{\partial(s)} se_i s^{-1} = \alpha_i e_1$, $\omega^{\partial(s)} se_i = \alpha_i e_i s$ for all i. Since $\omega^{\partial(s)}$ is independent of e_i, working analogously to the even case, we obtain the result.□

3. Norm and Admissible Norms of a Generalized Clifford Algebra

In this section we define a map $N : C(V) \to C(V)$ which we call the norm of $C(V)$ and which generalizes the spinor norm (see Lam, 1973, p. 109; Fröhlich, 1985, Appendix I).

Definition 3.1. For $V = \langle e_1, \ldots, e_m \rangle$, with $\{e_1, \ldots, e_m\}$ an admissible basis, we call $\beta : C(V) \longrightarrow C(V)$ the map defined by

$$\beta\left(\sum \alpha_{\varepsilon_1,\ldots,\varepsilon_m} e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}\right) = \sum \alpha^{n-1}_{\varepsilon_1,\ldots,\varepsilon_m} e_m^{\varepsilon_m(n-1)} \cdots e_1^{\varepsilon_1(n-1)}, \qquad \alpha_{\varepsilon_1,\ldots,\varepsilon_m} \in K.$$

Remark 3.2. (a) β is independent of the admissible basis. If $i < j$, $\beta(e_i e_j) = \beta(\omega e_j e_i)$.

(b) For $n = 2$, β is an antiautomorphism (see Fröhlich, 1985, Appendix I). For $n > 2$ we have that if x or y is an element of $C(V)$ of the form $\lambda e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}$ for some $\lambda \in K$ and $\varepsilon_i \in \{0, \ldots, n-1\}$, then $\beta(xy) = \beta(y)\beta(x)$.

Definition 3.3. We call the norm of the generalized Clifford algebra $C(V)$ the map $N : C(V) \longrightarrow C(V)$ given by $N(z) := \beta(z)z$.

By the properties of β, N is independent of the admissible base. Moreover, $N(\lambda) = \lambda^n$ if $\lambda \in K$, $N(e_i) = e_i^n = a_i \in K$ for each i and $N(e_i e_j) = N(e_i)N(e_j)$.

In general, N is not multiplicative. For this reason, we define the subset $F(C(V))$ of $C(V)^*$

$$F(C(V)) := \{x \in C(V)^* \text{ homog. s.t. } N(x) \in K^* \text{ and } \beta(xy) = \beta(y)\beta(x) \, \forall y \in C(V)\}.$$

It is easy to verify that $F(C(V))$ is a subgroup of $C(V)^*$ which, for $n = 2$, is the Clifford group (see, Bourbaki 1950). If $x \in F(C(V))$ and $y \in C(V)$, then $N(xy) = N(x)N(y)$. In particular, the norm N restricted to the subgroup $F(C(V))$ is multiplicative. Moreover, if $x = \lambda e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}$ for some $\lambda \in K$ and $\varepsilon_i \in \{0, \ldots, n-1\}$, then $x \in F(C(V))$. As a consequence of Proposition 2.3, $\varphi(F(C(V)))$ contains the automorphisms which leave the space V invariant, where φ is the morphism defined in Proposition 2.2.

Next, we define admissible norms of a generalized Clifford algebra. From their properties, we construct an important exact sequence for the study of embedding problems.

Definition 3.4. Let A be a subgroup of $F(C(V))$ such that $K^* \subset A$. An admissible norm in A is a map $\mathcal{N} : A \longrightarrow K^*$ such that for $a, a_1, a_2 \in A$,

$$\mathcal{N}(a) \in K^*, \qquad \mathcal{N}(a_1 a_2) = \mathcal{N}(a_1)\mathcal{N}(a_2) \qquad \text{and} \qquad \mathcal{N}(\lambda) = \lambda^n \text{ if } \lambda \in K.$$

In particular, for $A = F(C(V))$, the norm N is an admissible norm. If $\mathcal{N}$ is an admissible norm in $C(V)$, by restricting the morphism φ defined in Proposition 2.2 to the subgroup A, we have the following proposition.

Proposition 3.5. *The following diagram is commutative with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_n & \to & \mathrm{Ker}\mathcal{N} & \to & \mathrm{Ker}\mathcal{N}/\mu_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & K^* & \to & A & \xrightarrow{\varphi} & \varphi(A) & \to & 1 \\
 & & \downarrow & & \downarrow \mathcal{N} & & \downarrow P\mathcal{N} & & \\
1 & \to & {K^*}^n & \to & K^* & \to & K^*/{K^*}^n & \to & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

where $P\mathcal{N}$ is induced by $\mathcal{N}$ on $\varphi(A)$.

Proposition 3.6. *The sequence $1 \to \mu_n \simeq \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ker}\mathcal{N} \to \varphi(A)$ is exact where $\mathrm{Ker}\mathcal{N} \to \varphi(A)$ is the composition $\mathrm{Ker}\mathcal{N} \to A \xrightarrow{\varphi} \varphi(A)$ given by the above diagram.*

If L is a field extension of K, we have the exact sequence $1 \to \mu_n \to \mathrm{Ker}\mathcal{N}_L \to \varphi(A_L)$, where $\mathcal{N}_L$ denotes the extension of $\mathcal{N}$ to the subgroup A_L of $F(C(V) \otimes_K L)$ generated by A. For $L = K^{sep}$, we deduce the exact sequence

$$1 \longrightarrow \mu_n \to \mathrm{Ker}\mathcal{N}_{K^{sep}} \longrightarrow \varphi(A_{K^{sep}}) \longrightarrow 1.$$

4. Galois Symbols and Brauer Invariant

We now study properties of Galois symbols and we compute the Brauer invariant of some generalized Clifford algebras.

The definition of a Galois symbol is given in Serre (1968, XIV.2, Proposition 5) as follows. Given $a, b \in K^*$, let $\chi_a, \chi_b \in \mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z})$ be the corresponding elements by the Kummer isomorphism $K^*/K^{*^n} \simeq \mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z})$. The Galois symbol, denoted by (a, b), is

$$(a, b) = \chi_a \cup \chi_b \in H^2(G_K, \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{Br}_n(K)$$

where $\mathrm{Br}_n(K)$ is the n-torsion subgroup of the Brauer group and we consider the cup-product $\cup : H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \times H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Br}_n(K)$ via the multiplication in $\mathbb{Z}$.

Some properties of the Galois symbol are described in Serre (1968, XIV.2, Proposition 4). In particular the symbol $(a, b) = 0$ if and only if a is a norm in $K(\sqrt[n]{b})$ if and only if b is a norm in $K(\sqrt[n]{a})$. We deduce additional properties of Galois symbols in the following lemma.

Lemma 4.1. $(a, a) = (a, (-1)^{n-1})$ *and* $((-1)^{n-1}a, a) = (a, (-1)^{n-1}a) = 0$.

In Vela (1998a, Chapter 3), we define the Brauer invariant and study its behaviour under the graded tensor product.

Definition 4.2. Let A be a central simple graded algebra.

(a) If A is of even type, A is a central simple algebra and we define the Brauer invariant of A, $[A] \in \mathrm{Br}(K)$ as the class of A in the Brauer group as a central simple algebra.
(b) If A is of odd type, A_0 is a central simple algebra and we define the Brauer invariant of A, $[A] \in \mathrm{Br}(K)$, as the class of A_0 in the Brauer group.

Theorem 4.3. *Let A and A' be two central simple graded algebras with invariants a, a' respectively. The Brauer invariant of their graded tensor product is:*

(a) $[A\hat{\otimes}A'] = [A] + [A'] + (a, a')$ *if both are of odd type,*
(b) $[A\hat{\otimes}A'] = [A] + [A'] + (a, (-1)^{n-1}a')$ *if A is of even type and A' is of odd type,*
(c) $[A\hat{\otimes}A'] = [A] + [A'] + ((-1)^{n-1}a^{-1}, a^{-1}a')$ *if A is of odd type and A' is of even type,*
(d) $[A\hat{\otimes}A'] = [A] + [A'] + (a^{-1}, (-1)^{n-1}a^{-1}a')$ *if both are of even type.*

We now compute the Brauer invariant of some generalized Clifford algebras.

Proposition 4.4. *(a)* $[C(a)] = 0$.
(b) $[C(a_1, a_2)] = (a_1, a_2)$.
(c) $[C(a_1, a_2, a_3)] = (a_1, a_2) + (a_1^{-1}, a_3) + (a_2, a_3) + ((-1)^{n-1}, (-1)^{n-1}a_1a_2a_3)$.
(d) $[C(a_1, a_2, a_3, a_4)] = (a_1, a_2) + (a_1^{-1}, a_3) + (a_1, a_4) + (a_2, a_3) + (a_2^{-1}, a_4) + (a_3, a_4) + ((-1)^{n-1}, (-1)^{n-1}a_1a_2a_3a_4)$.

The second part is Soulé (1982, II.1). The others are consequences of properties of Galois symbols and Theorem 4.3.

Proposition 4.5. *Let $C(a_1,\dots,a_m)$ be a generalized Clifford algebra with invariant d and $b\in K^*$. Then, the invariant of $C(ba_1,\dots,ba_m)$ is d if m is even and bd if m is odd. Moreover,*

(a) $[C(ba_1,\dots,ba_m)] = [C(a_1,\dots,a_m)] + (b,(-1)^{\frac{(n-1)(m-2)}{2}} ba_1^{-1}a_2\cdots a_{m-1}^{-1}a_m)$ *if m is even,*
(b) $[C(ba_1,\dots,ba_m)] = [C(a_1,\dots,a_m)]$ *if m is odd.*

The first part is the calculation of the invariant from structure theorems. The second is proved by induction on m.

5. Representations and Associated Cohomology Classes. The Twisted Algebra by a Representation

In this section we study the representations of a profinite group in the graded automorphism group of a generalized Clifford algebra and some cohomology classes associated to it.

Definition 5.1. Let Γ be a profinite group. A representation of Γ over K is the pair given by

(1) a generalized Clifford algebra $C_t = C(V_t)$ over K with an admissible norm $(\mathcal{N}, A_t)$ together with
(2) a continuous homomorphism $t:\Gamma\to O(C_t)\subset \mathrm{Autgr}(C_t)$, where $O(C_t)=\varphi(A_t)$ with φ the homomorphism defined in Proposition 2.2.

If t is fixed we omit it in the notation.

We say that a representation *has degree* 0 or *is special* if for each $\sigma\in\Gamma$, the elements in $\varphi^{-1}(t(\sigma))$ have degree 0. It is clear that all the elements in $\varphi^{-1}(t(\sigma))$ have the same degree.

Definition 5.2. Given two representations t and t' of Γ over K, the graded tensor product of t and t', $t\hat{\otimes}t'$ is the representation of Γ where

(1) The associated generalized Clifford algebra is $C_{t\hat{\otimes}t'} = C_t\hat{\otimes}C_{t'}$,
the set $A_{t\hat{\otimes}t'} = \{x\otimes y \mid x\in A_t, y\in A_{t'}\}\cap F(C_{t\hat{\otimes}t'})\subset C_{t\hat{\otimes}t'}$
and the norm $\mathcal{N}_{t\hat{\otimes}t'}$ is $\mathcal{N}_{t\hat{\otimes}t'}(x\otimes y) = \mathcal{N}_t(x)\mathcal{N}_{t'}(y)$.
(2) The morphism for $\sigma\in\Gamma$ is
$$t\hat{\otimes}t'(\sigma) = \varphi_{C_{t\hat{\otimes}t'}}(s(\sigma)\otimes 1)(1\otimes s'(\sigma)) = \varphi_{C_{t\hat{\otimes}t'}}(s(\sigma)\otimes s'(\sigma))\in O(C_t\hat{\otimes}C_{t'})$$
where $s(\sigma)\in\varphi_{C_t}^{-1}(t(\sigma))\subset C_t$ and $s'(\sigma)\in\varphi_{C_{t'}}^{-1}(t'(\sigma))\subset C_{t'}$.

In fact, except for a root of unity that can be computed explicitly, $(t\hat{\otimes}t')(\sigma)(x\otimes y)$ is $t(\sigma)(x)\otimes t'(\sigma)(y)$.

Let t be a representation of Γ. We consider Γ to be acting trivially on $\mathbb{Z}/n\mathbb{Z}$. We now define some cohomology classes associated to t.

Definition 5.3. Let $t : \Gamma \to O(C_t)$ be a representation of Γ over K. Let $s(\sigma) \in \varphi^{-1}(t(\sigma))$ where φ is the morphism defined in Proposition 2.2.

(a) We call $d(t) \in \mathrm{Hom}(\Gamma, \mathbb{Z}/n\mathbb{Z}) = H^1(\Gamma, \mathbb{Z}/n\mathbb{Z})$ the element defined by

$$\begin{aligned} d(t) : \Gamma &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ \sigma &\longmapsto \partial(s(\sigma)) \text{ where } \partial \text{ denotes the degree.} \end{aligned}$$

(b) The element $P\mathcal{N}_2[t] \in H^2(G_K, \mathbb{Z}/n\mathbb{Z}) = Br_n(K)$ is the class of the cocycle

$$\begin{aligned} e_t : G_K \times G_K &\to \mathbb{Z}/n\mathbb{Z} \\ (\sigma, \tau) &\mapsto e_t(\sigma,\tau) = [P\mathcal{N}(t(\tau))](\sigma) = lg\frac{\sigma(\sqrt[n]{\mathcal{N}(s(\tau))})}{\sqrt[n]{\mathcal{N}(s(\tau))}}, \end{aligned}$$

where the last equality is obtained from the isomorphism of Kummer theory

$$K^*/{K^*}^n \simeq \mathrm{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) = H^1(G_K, \mathbb{Z}/n\mathbb{Z}).$$

From the exact sequence $1 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ker}\mathcal{N}_{K^{sep}} \to O(C_t \otimes_K K^{sep}) \to 1$ and composing the representation t with

$$O(C_t) \to O(C_t \otimes_K K^{sep}) \subset \mathrm{Autgr}(C_t \otimes_K K^{sep})$$

we obtain

$$\begin{array}{ccccccccc} & & & & & & \Gamma & & \\ & & & & & & \downarrow t & & \\ & & & & & & O(C_t) & & \\ & & & & & & \downarrow & & \\ 1 & \to & \mathbb{Z}/n\mathbb{Z} & \to & \mathrm{Ker}\mathcal{N}_{K^{sep}} & \to & O(C_t \otimes_K K^{sep}) & \to & 1 \end{array}$$

that is an extension of Γ by $\mathbb{Z}/n\mathbb{Z}$ and so, an element of $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$.

Definition 5.4. We call the analogue to the second Stiefel–Whitney class, denoted by s_t, the element of $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ given by the above construction.

If $\Gamma = G_K$, the element s_t belongs to $H^2(G_K, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Br}_n(K)$.

For $\sigma \in \Gamma$, we put $t(\sigma) \otimes Id \in O(C_t \otimes K^{sep})$ and we consider an element $t'(\sigma) \in \varphi^{-1}_{C_t \otimes K^{sep}}(t(\sigma) \otimes Id) \subset C_t \otimes K^{sep}$ such that $t'(\sigma) \in \mathrm{Ker}\mathcal{N}_{K^{sep}}$, that is, $\mathcal{N}_{K^{sep}}(t'(\sigma)) = 1$. This element exists because the sequence of Proposition 3.6 is exact. In fact $t'(\sigma) = s(\sigma) \otimes \lambda_\sigma$ where $\lambda_\sigma \in K^{sep}$. The map

$$b_t : \Gamma \times \Gamma \xrightarrow{b_{0t}} \mu_n \xrightarrow{\lg} \mathbb{Z}/n\mathbb{Z} \qquad \text{where} \qquad b_{0t}(\sigma,\tau) = t'(\sigma)t'(\tau)t'(\sigma\tau)^{-1}, \quad \sigma, \tau \in \Gamma,$$

is a 2-cocycle of $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ representing the element s_t.

We now study the behaviour of these cohomology classes under the tensor product of representations.

Proposition 5.5. *Let t_1 and t_2 be representations of Γ over K and $s_i(\sigma) \in \varphi^{-1}_{C_i}(t_i(\sigma))$ for $\sigma \in \Gamma$. For a representation t, we denote $b_t \in K^*/{K^*}^n$ the element corresponding to $d(t)$ by Kummer isomorphism. Then:*

(a) $d(t_1 \hat{\otimes} t_2) = \partial(s_1(\sigma)s_2(\sigma)) = \partial(s_1(\sigma)) + \partial(s_2(\sigma)),\ \sigma \in \Gamma$ *and* $b_{t_1 \hat{\otimes} t_2} = b_{t_1} b_{t_2}$.

(b) $P\mathcal{N}_2[t_1\hat{\otimes}t_2] = P\mathcal{N}_2[t_1] + P\mathcal{N}_2[t_2]$.
(c) $s_{t_1\hat{\otimes}t_2} = s_{t_1} + s_{t_2} - (d(t_2) \cup d(t_1)) = s_{t_1} + s_{t_2} + (b_{t_1}, b_{t_2})$.

Proof. (a) This is straightforward to check.
(b) $P\mathcal{N}[t_1\hat{\otimes}t_2](\sigma) = \mathcal{N}(s_1(\sigma)\cdot s_2(\sigma)) = P\mathcal{N}[t_1](\sigma)\cdot P\mathcal{N}[t_2](\sigma)$.
(c) By definition of $t_1\hat{\otimes}t_2$ we can consider for all $\sigma \in \Gamma$, $(t_1\hat{\otimes}t_2)'(\sigma) = t_1'(\sigma)t_2'(\sigma)$. Then

$$b_{0t_1}(\sigma,\tau)b_{0t_2}(\sigma,\tau)t_1'(\sigma\tau)t_2'(\sigma\tau) = t_1'(\sigma)t_1'(\tau)t_2'(\sigma)t_2'(\tau),$$

and we obtain

$$b_{t_1}(\sigma,\tau) + b_{t_2}(\sigma,\tau) = [d(t_2)\cup d(t_1)](\sigma,\tau) + b_{t_1\hat{\otimes}t_2}$$

because $t_1'(\tau)t_2'(\sigma) = \omega^{\partial(t_1'(\tau))\partial(t_2'(\sigma))}t_2'(\sigma)t_1'(\tau)$. Thus,

$$s_{t_1\hat{\otimes}t_2} = s_{t_1} + s_{t_2} - (d(t_2)\cup d(t_1)) = s_{t_1} + s_{t_2} + (b_{t_1}, b_{t_2}).$$

□

We can express the element $P\mathcal{N}_2[t]$ as a sum of Galois symbols. A similar computation for $n = 2$ is made in Fröhlich (1985, Proposition 7.8). For reduced norms in a central simple algebra there is an analogous computation in Crespo (1991, Proposition 1).

Proposition 5.6. *Let t be a representation of Γ over K. We consider the elements*

(1) $b_1,\dots,b_r \in K^*$ *independent modulo* ${K^*}^n$ *such that the fixed field of* $\mathrm{Ker}P\mathcal{N}[t] \subset \Gamma$ *is contained in* $K(\sqrt[n]{b_1},\dots,\sqrt[n]{b_r})$,
(2) $\sigma_1,\dots,\sigma_r \in \Gamma$ *satisfying* $\frac{\sigma_i(\sqrt[n]{b_j})}{(\sqrt[n]{b_j})} = \omega^{\delta_{ij}}$ *and*
(3) $a_1,\dots,a_r \in K^*$ *such that* $P\mathcal{N}[t](\sigma_i) \equiv a_i \pmod{{K^*}^n}$.

We then have

$$P\mathcal{N}_2[t] = \sum_{k=1}^{r}(a_k, b_k).$$

Proof. The 2-cocycle representing $P\mathcal{N}_2[t]$ is $e_t(\tau,\sigma)$. We consider, for $k = 1,\dots,r$, the elements $\theta_k(\sigma) = lg(\frac{\sigma(\sqrt[n]{b_k})}{\sqrt[n]{b_k}})$ and $\phi_k(\sigma) = lg(\frac{\sigma(\sqrt[n]{a_k})}{\sqrt[n]{a_k}}) \in \mathrm{Hom}(\Gamma,\mathbb{Z}/n\mathbb{Z})$ corresponding to b_k and a_k by Kummer isomorphism. We have $\theta_k(\sigma_i) = lg(\sigma_i(\sqrt[n]{b_k})(\sqrt[n]{b_k})^{-1}) = \delta_{ik}$. So that

$$e_t(\tau,\sigma_i) = \phi_i(\tau) = \sum_{k=1}^{r}\phi_k(\tau)\theta_k(\sigma_i).$$

Hence $e_t = \sum_{k=1}^{r}\phi_k \cup \theta_k$ and $P\mathcal{N}_2[t] = \sum_{k=1}^{r}(a_k, b_k)$.□

Next we define the twisted algebra by a representation. Let L/K be a Galois extension and $\Gamma = \mathrm{Gal}(L/K)$. We consider Γ acting trivially on $\mathbb{Z}/n\mathbb{Z}$.
Let $t : \Gamma \to O(C_t) \subset \mathrm{Autgr}(C_t)$ be a representation. We consider the morphism

$$\alpha : \Gamma \xrightarrow{t} O(C_t) \to O(C_t\otimes L) \subset \mathrm{Autgr}(C_t\otimes L), \qquad \sigma \mapsto t(\sigma)_L(x\otimes\lambda) = t(\sigma)(x)\otimes\lambda$$

which is an element of $H^1(\Gamma, \mathrm{Aut}(C_t \otimes L))$, because the elements $t(\sigma)_L$ are fixed by the action of Γ defined in Fröhlich (1985, III).

DEFINITION 5.7. We use the term twisted algebra of C_t by the representation t to refer to the K-algebra $\mathfrak{C}_t$ corresponding to the element $\alpha \in H^1(\Gamma, \mathrm{Aut}(C_t \otimes L))$ by the bijection given in Fröhlich (1985, III.2) (see also Serre, 1968, Chapter X).

We have the isomorphism $g : \mathfrak{C}_t \otimes L \xrightarrow{\sim} C_t \otimes L$, where Γ acts on $C_t \otimes L$ via $1 \otimes gal$, that is, $\sigma(x \otimes \lambda) = x \otimes \sigma(\lambda)$.

By the bijection given in Fröhlich (1985, III.2), we have $\mathfrak{C}_t \simeq (C_t \otimes L)^\Gamma$ where Γ acts on $C_t \otimes L$ via $t \otimes gal$, i.e., $\sigma(x \otimes \lambda) = t(\sigma)(x) \otimes \sigma(\lambda)$.

This morphism satisfies $g^{-1}g^\sigma = t(\sigma)$ for each $\sigma \in \Gamma$.

Since C_t is a central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebra, by the equivalence of categories given in Fröhlich (1985, III.1), $\mathfrak{C}_t$ is also a central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebra. Moreover, if C_t is odd (even) then $\mathfrak{C}_t$ is odd (even). We now compute a structure element and the invariant of $\mathfrak{C}_t$.

PROPOSITION 5.8. *Let t be a representation of Γ. Let z be a structure element of C_t and a its invariant. We consider $b \in K^*/K^{*^n}$ the element corresponding to $d(t)$ by Kummer isomorphism. Then $z \otimes \sqrt[n]{b^{-1}}$ is a structure element and ab^{-1} is the invariant of $\mathfrak{C}_t$.*

In particular, if t is a representation of degree 0, the element b is equal to 1 and C_t and $\mathfrak{C}_t$ have the same invariant.

The proof is consequence of the characterization of the structure elements given in Vela (1998a).

6. Relation Between the Cohomology Classes Associated to a Representation

We now suppose $L = K^{sep}$ and $G_K = \mathrm{Gal}(L/K)$. Let $t : G_K \to O(C_t)$ be a representation of G_K, where C_t is a generalized Clifford K-algebra with an admissible norm $\mathcal{N}$. We find a formula to relate the cohomology classes defined in Definition 5.3 with the Brauer invariants of C_t and $\mathfrak{C}_t$. From the exact sequence

$$1 \to \mu_n \to \mathrm{Ker}\mathcal{N}_{K^{sep}} \to O(C_t \otimes K^{sep}) \to 1,$$

we consider the long exact sequence of cohomology and the connecting morphism

$$\delta : H^1(G_K, O(C_t \otimes L)) \longrightarrow H^2(G_K, \mathbb{Z}/n\mathbb{Z}).$$

THEOREM 6.1. *Let t be a representation of G_K. Then*

$$\delta(\alpha) = s_t - P\mathcal{N}_2[t] \in H^2(G_K, \mathbb{Z}/n\mathbb{Z}),$$

for $\alpha \in H^1(G_K, O(C_t \otimes L))$ defined before.

PROOF. For each $\sigma \in G_K$, the element $t(\sigma) \otimes Id \in O(C_t \otimes L)$. For $s(\sigma) \in \varphi^{-1}(t(\sigma))$, conjugation by $s(\sigma) \otimes 1$ is the automorphism $t(\sigma) \otimes Id$. By the previous short exact sequence, there exists an element $t'(\sigma) \in C_t \otimes L$ such that conjugation by $t'(\sigma)$ is $t(\sigma) \otimes Id$, and $\mathcal{N}_{K^{sep}}(t'(\sigma)) = 1$. In fact, $t'(\sigma) = s(\sigma) \otimes \lambda_\sigma$ where $\lambda_\sigma \in L$ and $\lambda_\sigma^{-n} = P\mathcal{N}_K(t(\sigma))$.

We consider the 2-cocycle $a_t(\sigma,\tau) = lg(t'(\sigma)\sigma(t'(\tau))t'(\sigma\tau)^{-1})$ of G_K with values in $\mathbb{Z}/n\mathbb{Z}$. By Serre (1968, Chapter VII, Annexe) the class of a_t in $H^2(G_K,\mathbb{Z}/n\mathbb{Z})$ is $\delta(\alpha)$. Since

$$t'(\sigma)\sigma(t'(\tau))t'(\sigma\tau)^{-1}\sigma(\sqrt{P\mathcal{N}_K(t(\sigma))})^{-1}\sqrt{P\mathcal{N}_K(t(\sigma))} = t'(\sigma)t'(\tau)t'(\sigma\tau)^{-1},$$

we obtain the equality $a_t(\sigma,\tau) + e_t(\sigma,\tau) = b_t(\sigma,\tau)$ for all $\sigma,\tau \in G_K$ (where e_t, b_t are defined in Definition 5.3), and so $\delta(\alpha) + P\mathcal{N}_2[t] = s_t \in H^2(G_K,\mathbb{Z}/n\mathbb{Z})$.□

Proposition 6.2. *Suppose t is a representation of degree 0 or C_t is even. Let $\mathfrak{C}_t$ be the twisted algebra of C_t by t. Then*

$$[\mathfrak{C}_t] - [C_t] = \delta(\alpha) = s_t - P\mathcal{N}_2[t],$$

where $\alpha \in H^1(G_K, O(C_t \otimes L))$ is the element defined before and which corresponds to $\mathfrak{C}_t$ by the bijection given in Fröhlich (1985, III.2).

Proof. It is only necessary to keep in mind the isomorphism $\mathrm{Br}(K) \simeq H^2(G_K, K^{sep})$ proved in Serre (1968, X.5).□

We now consider the case when the representation t does not have degree 0. From t, we construct another representation t_0 with degree 0 and we apply to it the above result. Let t be a representation and let m be the number of generators of C_t as K-algebra. We define the generalized Clifford algebra $C_{\overline{t}} = C_1^{(n)}(e_{m+1})$ where $e_{m+1}^n = 1$ and the representation

$$\overline{t} : G_K \longrightarrow \mathrm{Autgr}(C_{\overline{t}}), \qquad \sigma \mapsto \varphi_{C_{\overline{t}}}(e_{m+1}^{n-\partial(s(\sigma))}) \qquad \text{where } s(\sigma) \in \varphi_{C_t}^{-1}(t(\sigma)).$$

Let t_0 be the representation $t\hat{\otimes}\overline{t}$. By definition, the generalized Clifford algebra associated to t_0 is the algebra $C_{t_0} = C_{t\hat{\otimes}\overline{t}} = C_t\hat{\otimes}C_{\overline{t}}$. The representation $t_0 : G_K \to \mathrm{Autgr}(C_{t_0})$ has degree 0.

Formula 6.3. *Let a_t be the invariant of C_t and b_t the element corresponding to $d(t)$ by Kummer theory. Then*

(a) $[\mathfrak{C}_{t_0}] - [C_t] + (a_t,a_t) = s_t + (b_t,b_t) - P\mathcal{N}_2[t]$ *if m is even,*
(b) $[\mathfrak{C}_{t_0}] - [C_t] = s_t + (b_t,b_t) - P\mathcal{N}_2[t]$ *if m is odd.*

Proof. Applying Proposition 6.2 to t_0

$$[\mathfrak{C}_{t_0}] - [C_{t_0}] = s_t + s_{\overline{t}} + (b_t, b_{\overline{t}}) - (P\mathcal{N}_2[t] + P\mathcal{N}_2[\overline{t}])$$

where $b_t, b_{\overline{t}} \in K^*/K^{*^n}$ are the elements corresponding to $d(t)$ and $d(\overline{t})$ respectively by Kummer theory. It is simple to check that $s_{\overline{t}} = 0$ and $P\mathcal{N}_2[\overline{t}] = 0$. The class $[C_{\overline{t}}] = 0$ and its invariant is $\overline{a_t} = 1$. Using Theorem 4.3 we obtain that if m is even $[C_{t_0}] = [C_t]+(a_t,a_t)$ and if m is odd $[C_{t_0}] = [C_t]$. We have $b_t^{-1} \equiv b_{\overline{t}} \pmod{K^{*^n}}$ by definition of $d(\overline{t})$. Thus $(b_t, b_{\overline{t}}) = -(b_t,b_t) = (b_t,b_t)$.□

Remark 6.4. If n is odd, $(a_t,a_t) = 0$ and $(b_t,b_t) = 0$. Then $[\mathfrak{C}_{t_0}] - [C_t] = s_t - P\mathcal{N}_2[t]$.

To simplify the above formula we have to study $\mathfrak{C}_{t_0}$. It is not true that $\mathfrak{C}_{t_0} \simeq \mathfrak{C}_t\hat{\otimes}\mathfrak{C}_{\overline{t}}$ where $\mathfrak{C}_{\overline{t}}$ is the twisted algebra corresponding to $\overline{t}$, but in some cases we can relate $\mathfrak{C}_{t_0}$ and $\mathfrak{C}_t$.

Proposition 6.5. *Suppose the twisted algebra of C_t by the representation t is $\mathfrak{C}_t = C_m(e_1 \otimes \lambda_1, \ldots, e_m \otimes \lambda_m)$. Let $a_i = e_i^n$ and b_t be the element corresponding to $d(t)$ by Kummer theory. Then*

(a) If m is even

$$\begin{aligned}\mathfrak{C}_{t_0} &= C\left(e_1 \otimes \lambda_1 \sqrt[n]{b_t^{-1}}, \ldots, e_m \otimes \lambda_m \sqrt[n]{b_t^{-1}}, e_{m+1} \otimes \sqrt[n]{b_t^{-1}}\right)\\ &= C(a_1\lambda_1^n b_t^{-1}, \ldots, a_m\lambda_m^n b_t^{-1}, b_t^{-1})\end{aligned}$$

and in this case $[\mathfrak{C}_{t_0}] = [\mathfrak{C}_t] + (a_t, a_t) + (b_t, b_t)$.

(b) If m is odd

$$\mathfrak{C}_{t_0} = C\left(e_1 \otimes \lambda_1, \ldots, e_m \otimes \lambda_m, e_{m+1} \otimes \sqrt[n]{b_t^{-1}}\right) = C(a_1\lambda_1^n, \ldots, a_m\lambda_m^n, b_t^{-1})$$

and $[\mathfrak{C}_{t_0}] = [\mathfrak{C}_t] + (b_t, b_t) - (a_t, b_t)$.

Proof. The element $e_{m+1} \otimes \sqrt[n]{b_t^{-1}}$ is fixed by the action $t_0 \otimes gal$, so belongs to $\mathfrak{C}_{t_0}$. In the same way, if $x \in C_t$ is an homogeneous element such that $x \otimes \lambda \in \mathfrak{C}_t$, we have $x \otimes \lambda \sqrt[n]{b_t^{-\partial(x)}} \in \mathfrak{C}_{t_0}$ if m is even and $x \otimes \lambda \in \mathfrak{C}_{t_0}$ if m is odd because it is fixed by $t_0 \otimes gal$. To compute the classes of the algebra it suffices to apply Theorem 4.5 and bear in mind structure theorems for the invariants of generalized Clifford algebras.□

Formula 6.6. *If $\mathfrak{C}_t = C_m(e_1 \otimes \lambda_1, \ldots, e_m \otimes \lambda_m)$ is the twisted algebra by the representation t,*

(a) $[\mathfrak{C}_t] - [C_t] = s_t - P\mathcal{N}_2[t]$ *if m is even,*
(b) $[\mathfrak{C}_t] - [C_t] - (a_t, b_t) = s_t - P\mathcal{N}_2[t]$ *if m is odd.*

We note that in the even case we recover the known formula.

Let L/K be a Galois extension, $\Gamma = \mathrm{Gal}(L/K)$ and $j : G_K \to \Gamma$ be the surjective homomorphism of Galois groups corresponding to L/K. We consider the homomorphism $\tilde{j} : \{\text{representations of } \Gamma \text{ over } K\} \to \{\text{representations of } G_K \text{ over } K\}$ where $\tilde{j}(t) = t \circ j$, and the homomorphisms between the cohomology groups, in particular

$$j_2^* : H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to H^2(G_K, \mathbb{Z}/n\mathbb{Z}).$$

We now have $C_{t\circ j} = C_t$, $\mathfrak{C}_{t\circ j} = \mathfrak{C}_t$ and $s_{t\circ j} = j_2^*(s_t)$.

Formula 6.7. *Let $t : \Gamma \to O(C_t)$ be a representation. If the hypothesis of Formula 6.6 holds, we have*

(a) $[\mathfrak{C}_t] - [C_t] = j_2^*(s_t) - P\mathcal{N}_2[t \circ j]$ *if m is even,*
(b) $[\mathfrak{C}_t] - [C_t] - (a_t, b_{t\circ j}) = j_2^*(s_t) - P\mathcal{N}_2[t \circ j]$ *if m is odd.*

7. Obstructions to the Solvability of Embedding Problems

We now use the results of previous sections to compute obstructions to the solvability of embedding problems given by central extensions with cyclic kernel in terms of Galois symbols. The theory of representations studied before is relevant if we can find a representation t of Γ so that $\varepsilon = s_t$. In this case, applying Formula 6.7 we can express $j^*(\varepsilon) = j^*(s_t)$ as a sum of Galois symbols. We find adequate representations for some examples.

7.1. **Example 1:** $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

For our first example, we take the field $L = K(\sqrt[n]{a})$ with degree $[L:K] = n$ and $\Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$. Let σ be the generator of Γ such that $\sigma(\sqrt[n]{a}) = \omega\sqrt[n]{a}$.

Theorem 7.1. *Consider the embedding problem given by*

$$L = K(\sqrt[n]{a})/K,\ \Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$$

and the exact sequence $(E) : 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{h} \mathbb{Z}/n\mathbb{Z} \to 0$, *where for* g, *a generator of* $\mathbb{Z}/n^2\mathbb{Z}$, *the morphisms are* $h(g) = \sigma$ *and* $g^n = i(lg(\omega))$.

Then, the obstruction to the solvability is (a, ω).

By the properties of Galois symbols, the embedding problem considered is solvable if and only if $(a,\omega) = 0$. This is equivalent to ω is a norm in $K(\sqrt[n]{a})/K$ or a is a norm in $K(\sqrt[n]{\omega})/K$.

Proof. Let $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ be the class corresponding to (E). Let j be the surjective morphism $j : G_K \to \Gamma$. We look for a representation t of Γ such that $\varepsilon = s_t$. We consider the representation of degree 0, $t : \Gamma \to O(C_t), \sigma \mapsto \varphi(e_1^{n-1}e_2)$, where $C_t = C(1, (-1)^{n-1}\omega)$ is the generalized Clifford algebra generated by the elements e_1, e_2 with the relations $e_1^n = 1$, $e_2^n = (-1)^{n-1}\omega$, and the admissible norm $\mathcal{N} : A_t \longrightarrow K^*$ given by $\mathcal{N}(e_1) = \mathcal{N}(e_2) = 1$ for the set $A_t = \{x = \lambda e_1^{\varepsilon_1}e_2^{\varepsilon_2} \in C_t^* \mid \lambda \in K, 0 \le \varepsilon_i \le n-1\} \subset F(C_t)$.

The element $(e_1^{n-1}e_2)^n = \omega$, so $e_1^{n-1}e_2$ is an element of order n^2 and its norm is $\mathcal{N}(e_1^{n-1}e_2) = 1$. Taking $t'(\sigma) = e_1^{n-1}e_2$ (with the notations of previous sections) and denoting $\langle t'(\sigma)\rangle$ the group $\varphi^{-1}(t(\Gamma)) \cap \mathrm{Ker}\mathcal{N}_{K^{sep}}$ the exact sequences

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \langle t'(\sigma)\rangle \to \Gamma \to 0 \qquad \text{and} \qquad 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{h} \Gamma \to 0$$

are equivalent and so $\varepsilon = s_t$. Applying Formula 6.7 for a representation of degree 0 we have $[\mathfrak{C}_t] - [C_t] = j_2^*(s_t) - P\mathcal{N}_2[t \circ j]$. Of these terms $[C_t] = (1, (-1)^{n-1}\omega) = 0$ and $P\mathcal{N}_2[t \circ j] = 0$ because $\mathcal{N}((e_1^{n-1}e_2)^r) = 1$, for all $r \in \{0, \ldots, n-1\}$. The twisted algebra is $\mathfrak{C}_t = C(e_1 \otimes \sqrt[n]{a}, e_2 \otimes \sqrt[n]{a}) = C(a, (-1)^{n-1}\omega a)$ because the elements $e_1 \otimes \sqrt[n]{a}$ and $e_2 \otimes \sqrt[n]{a}$ are fixed by the action $t \otimes gal$ in Γ. Therefore $[\mathfrak{C}_t] = (a, (-1)^{n-1}\omega a) = (a, (-1)^{n-1}a) + (a,\omega) = (a,\omega)$. We then obtain $j_2^*(s_t) = (a,\omega)$.□

7.2. **Example 2:** $0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to 0$

We now take $L = K(\sqrt[n]{a_1}, \sqrt[n]{a_2})$ such that $[K(\sqrt[n]{a_1}) : K] = [K(\sqrt[n]{a_2}) : K] = n$, $[L:K] = n^2$ and $\Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Let h_1 and h_2 be generators of Γ determined by $h_i(\sqrt[n]{a_j}) = \omega^{\delta_{ij}}\sqrt[n]{a_j}$ (where δ_{ij} is the δ of Kronecker). We want to compute the obstruction to the solvability of the embedding problems given by

$$L/K, \Gamma, (E) : 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} E \xrightarrow{h} \Gamma \to 0$$

where E is a group with n^3 elements.

There are different groups E that are extensions of Γ by $\mathbb{Z}/n\mathbb{Z}$. If E is Abelian, it is isomorphic to $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n^2\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$. In these cases the obstruction is known. The first one is the same as Example 1. The second one corresponds to the trivial extension, so the obstruction is trivial. We now compute the obstruction for two non-Abelian groups of order n^3.

THEOREM 7.2. *Let us consider the embedding problem given by*

$$L = K(\sqrt[n]{a_1},\sqrt[n]{a_2})/K \qquad \text{with } \Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$$

and the exact sequence $0\to\mathbb{Z}/n\mathbb{Z}\xrightarrow{i}E_1\xrightarrow{h}\Gamma\to 0$ *where* E_1 *is the group*

$$E_1 = \langle \sigma,\tau \mid \sigma \text{ has order } n, \tau \text{ has order } n^2, \sigma\tau = \tau^{n+1}\sigma\rangle,$$

and the morphisms are given by $h(\sigma)=h_1, h(\tau)=h_2$ *and* $i(lg(\omega))=\tau^n$.

Then, the obstruction to its solvability is $(\omega^{-1}a_1,a_2)$.

PROOF. Let $\varepsilon\in H^2(\Gamma,\mathbb{Z}/n\mathbb{Z})$ be the class corresponding to the exact sequence. Let j be the surjective morphism $j: G_K\to\Gamma$. We look for a representation t of Γ such that $\varepsilon = s_t$. We consider the morphism $t:\Gamma\to O(C_t), h_1\mapsto\varphi(e_1), h_2\mapsto\varphi(e_2)$ where $C_t = C(e_1,e_2)=C(1,\omega)$ is the generalized Clifford algebra with the admissible norm defined by $\mathcal{N}(e_1)=1$ and $\mathcal{N}(e_2)=1$ for the set $A_t=\{x=\lambda e_1^{\varepsilon_1}e_2^{\varepsilon_2}\in C_t^*\mid\lambda\in K, 0\le\varepsilon_i\le n-1\}\subset F(C_t)$.

We can take $t'(h_1)=e_1$ and $t'(h_2)=e_2$. The element $t'(h_1)$ has order n, $t'(h_2)$ has order n^2 and they satisfy the relations $t'(h_1)t'(h_2)=t'(h_2)^{n+1}t'(h_1)$. Therefore, if $\langle t'(h_1),t'(h_2)\rangle$ is the group $\varphi^{-1}(t(\Gamma))\cap\mathrm{Ker}\mathcal{N}_{K^{sep}}$, the exact sequences

$$0\to\mathbb{Z}/n\mathbb{Z}\to\langle t'(h_1),t'(h_2)\rangle\to\Gamma\to 0 \qquad\text{and}\qquad 0\to\mathbb{Z}/n\mathbb{Z}\xrightarrow{i}E_1\xrightarrow{h}\Gamma\to 0$$

are equivalent and $\varepsilon=s_t$. Since C_t has even type, Formula 6.7 is $[\mathfrak{C}_t]-[C_t]=j_2^*(s_t)-P\mathcal{N}_2[t\circ j]$. In this case $[C_t]=(1,\omega)=0$ and $P\mathcal{N}_2[t\circ j]=0$ because the elements of A_t have norm 1. Furthermore, the elements $e_1\otimes\sqrt[n]{a_2}, e_2\otimes\sqrt[n]{a_1^{-1}}\in\mathfrak{C}_t$ because they are fixed by $t\otimes gal$. Hence $\mathfrak{C}_t = C\left(e_1\otimes\sqrt[n]{a_2}, e_2\otimes\sqrt[n]{a_1^{-1}}\right)=C(a_2,\omega a_1^{-1})$ and $[\mathfrak{C}_t]=(a_2,\omega a_1^{-1})$.

The obstruction is then $j_2^*(s_t)=[\mathfrak{C}_t]=-(\omega a_1^{-1},a_2)=(\omega^{-1}a_1,a_2)$.□

REMARK 7.3. The choice of the morphisms in the exact sequence implies that, if the embedding problem is solvable and M is a solution field, then the extensions $M/K(\sqrt[n]{a_1})$ and $M/K(\sqrt[n]{a_1a_2})$ are cyclic while $M/K(\sqrt[n]{a_2})$ is not. Exchanging a_1 and a_2, or, equivalently, changing the morphisms of the exact sequence (E) so that $h(\sigma)=h_2$ and $h(\tau)=h_1$, the obstruction is $(a_1,\omega a_2^{-1})$. If the problem is solvable and M is a solution, the extension $M/K(\sqrt[n]{a_1})$ is not cyclic. Similarly, replacing a_2 by a_1a_2 and keeping a_1, the obstruction is $(\omega^{-1}a_1,a_1a_2)$ and the extension $M/K(\sqrt[n]{a_1a_2})$ is not cyclic.

THEOREM 7.4. *Let us consider the embedding problem given by*

$$L = K(\sqrt[n]{a_1},\sqrt[n]{a_2})/K \qquad \text{with } \Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$$

and the exact sequence $0\to\mathbb{Z}/n\mathbb{Z}\xrightarrow{i}E_2\xrightarrow{h}\Gamma\to 0$ *where* E_2 *is the group*

$$E_2 = \langle\sigma,\tau,\rho\mid\sigma,\tau,\rho \text{ have order } n,\ \sigma\tau\sigma^{-1}\tau^{-1}=\rho, \sigma\rho=\rho\sigma, \tau\rho=\rho\tau\rangle,$$

and the morphisms are given by $h(\sigma) = h_1, h(\tau) = h_2, h(\rho) = 1$ *and* $i(lg(\omega)) = \rho$. *Then, the obstruction to its solvability is* (a_1, a_2).

PROOF. As before, let $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ be the class corresponding to the exact sequence and $j : G_K \to \Gamma$. In this case, we consider $t : \Gamma \to O(C_t)$, $h_1 \mapsto \varphi(e_1)$, $h_2 \mapsto \varphi(e_2)$ for the generalized Clifford algebra $C_t = C(e_1, e_2) = C(1, 1)$, the set $A_t = F(C_t)$ and as admissible norm, the norm of the algebra N. One can easily verify that $\varepsilon = s_t$ and then $[\mathfrak{C}_t] - [C_t] = j_2^*(s_t) - P\mathcal{N}_2[t \circ j]$ with $[C_t] = (1, 1) = 0$ and $P\mathcal{N}_2[t \circ j] = 0$. In this case $\mathfrak{C}_t = C\left(e_1 \otimes \sqrt[n]{a_2}, e_2 \otimes \sqrt[n]{a_1^{-1}}\right) = C(a_2, a_1^{-1})$ and $[\mathfrak{C}_t] = (a_1, a_2)$. Therefore, the obstruction is $j_2^*(s_t) = [\mathfrak{C}_t] = (a_1, a_2)$.□

REMARK 7.5. If $n = p$ a prime integer, there are no other noncommutative groups of order n^3 different from E_1 and E_2. The group E_2 for $n = p$ prime is the Heisenberg group of degree p.

8. Explicit Construction of Solutions

In this section we give a method to solve explicitly Galois embedding problems given by central extensions with cyclic kernel. If $\lambda \in L$, we denote by λ^σ the action (on the left) of $\sigma \in \Gamma$ on λ. We shall prove equivalent conditions to the existence of solutions.

Let L/K a Galois extension with $\Gamma = \mathrm{Gal}(L/K)$ and consider the embedding problem

$$L/K,\, \Gamma,\, (E) : 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} E \xrightarrow{h} \Gamma \to 0,$$

given by a central extension of Γ with cyclic kernel. Let $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ be the element corresponding to (E).

PROPOSITION 8.1. *(a) Let* $\gamma \in L^*$ *be such that* $[L(\sqrt[n]{\gamma}) : L] = n$ *and the group* $\mathrm{Gal}(L(\sqrt[n]{\gamma})/L) \simeq \mathbb{Z}/n\mathbb{Z}$. *Then* $L(\sqrt[n]{\gamma})/K$ *is a solution to the embedding problem if and only if* $\gamma^\sigma = b_\sigma^n \gamma$, *for all* $\sigma \in \Gamma$ *where* $b_\sigma \in L^*$ *are such that* $(\sigma, \tau) \mapsto \lg(b_\sigma b_\tau^\sigma b_{\sigma\tau}^{-1})$ *is a 2-cocycle representing* ε.

(b) Given elements $\{b_\sigma\}_{\sigma \in \Gamma} \in L^*$ *such that* $(\sigma, \tau) \mapsto \lg(b_\sigma b_\tau^\sigma b_{\sigma\tau}^{-1})$ *is a 2-cocycle representing* ε, *then, there exists* $\gamma \in L^*$ *such that* $\gamma^\sigma = b_\sigma^n \gamma$ *for all* $\sigma \in \Gamma$. *In this case,* $L(\sqrt[n]{\gamma})/K$ *is a solution to the embedding problem.*

PROOF. *(a)* If $L(\sqrt[n]{\gamma})/L$ is a solution to the embedding problem, a 2-cocycle representing the exact sequence $0 \to \mathrm{Gal}(L(\sqrt[n]{\gamma})/L) \to \mathrm{Gal}(L(\sqrt[n]{\gamma})/K) \to \mathrm{Gal}(L/K) \to 0$ also represents (E). Let $b_\sigma \in L^*$ such that $\gamma^\sigma = b_\sigma^n \gamma$ for $\sigma \in \Gamma$. Taking a system of representatives $\{u_\sigma\}$ of Γ in $\mathrm{Gal}(L(\sqrt[n]{\gamma})/L)$ with $u_\sigma(\sqrt[n]{\gamma}) = b_\sigma \sqrt[n]{\gamma}$ and $a_{\sigma,\tau}$ such that $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$ it is straightforward to verify that $b_\tau^\sigma b_\sigma = \omega^{a_{\sigma,\tau}} b_{\sigma\tau}$.

The other implication is clear.

(b) We consider $\gamma = \sum_{\sigma \in \Gamma} b_\sigma^{-n} x^\sigma \in L$ for $x \in L^*$. For every $\sigma \in \Gamma$, we compute the relation

$$\gamma^\sigma = \sum_{\tau \in \Gamma} (b_\tau^{-n})^\sigma (x^\tau)^\sigma = b_\sigma^n \sum_{\tau \in \Gamma} b_{\sigma\tau}^{-n} x^{\sigma\tau} = b_\sigma^n \gamma.$$

By the theorem of linear independence of automorphisms there exists $x \in L^*$ such that $\gamma \neq 0$.□

We now give the main theorem about the explicit construction of solutions using our study of representations.

THEOREM 8.2. *Let L/K be a Galois extension and $\Gamma = \mathrm{Gal}(L/K)$. We consider the embedding problem given by $L/K, \Gamma, (E)$ where*

$$(E) : 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} E \xrightarrow{h} \Gamma \to 0.$$

Let $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ be the element corresponding to (E). Here we denote $C_L = C \otimes_K L$, $\mathfrak{C}$ the twisted algebra of C by the representation t and $\mathfrak{C}_L = \mathfrak{C} \otimes_K L$.

Suppose the following conditions hold:

(1) The embedding problem is solvable.
(2) There exists a representation (C, t) of degree 0 of Γ over K such that $s_t = \varepsilon$.
(3) For every $\sigma \in \Gamma$, there exists an element $u_\sigma \in F(C_L)$ such that $\varphi_{C_L}(u_\sigma) = t(\sigma)$ and $a_{\sigma,\tau} := u_\sigma u_\tau u_{\sigma\tau}^{-1}$ represents ε. (For a field large enough K', elements $u_\sigma \in C_{K'}$ such that $a_{\sigma,\tau} = u_\sigma u_\tau u_{\sigma\tau}^{-1}$ represents ε and $t(\sigma) = \varphi_{C_{K'}}(u_\sigma)$ exist because $s_t = \varepsilon$. Our hypothesis is that such elements can be found into $F(C_L)$.)

Let $g : C_L \to \mathfrak{C}_L$ be the graded isomorphism over L such that $g^{-1}g^\sigma = t(\sigma)$ for all $\sigma \in \Gamma$. Then, there exists a graded isomorphism $f : C \to \mathfrak{C}$ over K such that the homogeneous element of degree 0

$$z = \sum_{\epsilon_i \in \{0,\dots,n-1\}} g(e_1)^{\epsilon_1} g(e_2)^{\epsilon_2} \cdots g(e_m)^{\epsilon_m} f(e_m)^{-\epsilon_m} \cdots f(e_1)^{-\epsilon_1} \in \mathfrak{C}_L,$$

where $\{e_1, \dots, e_m\}$ is an admissible basis of the vector space V of C, is invertible.

Assume further that $n_\sigma := N(g(u_\sigma)) \in L$ and that there exists an element $\eta \in L$ such that $\eta^{-\sigma}\eta = n_\sigma^{-1}$ for all $\sigma \in \Gamma$.

Then, for $\alpha \neq 0$ be a coordinate of $N(z)$ in a Γ-invariant basis of $\mathfrak{C}_L$ and $\gamma = \eta\alpha^{-1}$, the field $L(\sqrt[n]{\gamma})$ is a solution to the embedding problem considered.

PROOF. Since the embedding problem is solvable, by Proposition 8.1 there exist $b_\sigma \in L^*$ such that $b_\sigma b_\tau^\sigma b_{\sigma\tau}^{-1} = \omega^{a_{\sigma,\tau}}$ for $a_{\sigma,\tau}$ a 2-cocycle representing ε. Then $\sigma \mapsto h_\sigma = b_\sigma^{-1} u_\sigma$ defines a 1-cocycle of Γ in $(C_L)_0^*$. By Serre (1968, X.1 Example 2), $H^1(\Gamma, (C_L)_0^*) = 0$, and so h_σ is a 1-coboundary: that is, there exists an element $c \in (C_L)_0^*$ such that $h_\sigma = cc^{-\sigma}$. Computing

$$(g^{-1}g^\sigma)(e_i) = t(\sigma)(e_i) = u_\sigma e_i u_\sigma^{-1} = cc^{-\sigma} e_i c^\sigma c^{-1} \Rightarrow (g^{-1} \circ g^\sigma)(x) = cc^{-\sigma} x c^\sigma c^{-1} \, \forall x \in C_L.$$

Putting $y = g^\sigma(x) \in \mathfrak{C}_L$, we have $g^{-1}(y) = cc^{-\sigma} g^{-\sigma}(y) c^\sigma c^{-1}$ and then $c^{-1} g^{-1}(y) c = c^{-\sigma} g^{-\sigma}(y) c^\sigma, \sigma \in \Gamma$.

This implies that $\Psi : \mathfrak{C}_L \to C_L$, defined by $\Psi(y) = c^{-1} g^{-1}(y) c$, sends Γ-invariant elements to Γ-invariant elements and is thus restricted to an isomorphism from $\mathfrak{C}$ to C. We now prove by induction on m that we can choose f such that z is different from 0.

If $m = 1$, we have $z = \sum_{i=0}^{n-1} g(e_1)^i f(e_1)^{-i}$.

If $z = 0$, let $f_k(e_1) = \omega^k f(e_1)$ for $0 \leq k \leq n-1$. We note that f_k is a graded isomorphism

too. Let z_k be the elements z corresponding to f_k, that is, $z_k = \sum_{i=0}^{n-1} g(e_1)^i f_k(e_1)^{-i}$. If, for some $k \in \{0, \dots, n-1\}$, $z_k \neq 0$ then we have finished.
If $z_k = 0$ for all k, we reach a contradiction because $\mathrm{char} K \nmid n$ and

$$0 = \sum_{k=0}^{n-1} z_k = \sum_{k=0}^{n-1} 1 + \sum_{i=1}^{n-1} \left(\sum_{k=0}^{n-1} \omega^{-ik} \right) g(e_1)^i f(e_1)^{-i} = \sum_{k=0}^{n-1} 1 = n.$$

Suppose the result is true for $m-1$ and consider

$$z = \sum_{\epsilon_i \in \{0,\dots,n-1\}} g(e_1)^{\epsilon_1} \cdots g(e_m)^{\epsilon_m} f(e_m)^{-\epsilon_m} \cdots f(e_1)^{-\epsilon_1}.$$

If $z = 0$, like before, we consider new graded isomorphisms $f_k(e_1) = \omega^k f(e_1)$ and $f_k(e_i) = f(e_i)$, $i \neq 1$. Let z_k be the elements corresponding to f_k.
If $z_k \neq 0$ for some k we have finished. If not, we consider the element

$$x_1 = \sum_{0 \le \epsilon_i \le n-1} g(e_2)^{\epsilon_2} \cdots g(e_m)^{\epsilon_m} f(e_m)^{-\epsilon_m} \cdots f(e_2)^{-\epsilon_2}.$$

We can express $z_k = \sum_{i=0}^{n-1} \omega^{-ki} g(e_1)^i x_1 f(e_1)^{-i}$. As before $0 = \sum_{k=0}^{n-1} z_k = n x_1$, then $x_1 = 0$ and we can apply the induction hypothesis.

We now prove that if $z \neq 0$ then it is invertible. We note that, if $j < i$, $f(e_j)^{-r} f(e_i)^{-1} = \omega^r f(e_i)^{-1} f(e_j)^{-r}$. Then

$$g(e_i) g(e_1)^{\epsilon_1} \cdots g(e_m)^{\epsilon_m} = \omega^{-(\epsilon_1 + \cdots + \epsilon_{i-1})} g(e_1)^{\epsilon_1} \cdots g(e_i)^{\epsilon_i + 1} \cdots g(e_m)^{\epsilon_m}$$

and

$$f(e_m)^{-\epsilon_m} \cdots f(e_1)^{-\epsilon_1} f(e_i)^{-1} = \omega^{(\epsilon_1 + \cdots + \epsilon_{i-1})} f(e_m)^{-\epsilon_m} \cdots f(e_i)^{-\epsilon_i - 1} \cdots f(e_1)^{-\epsilon_1}.$$

Hence, $g(e_i) z f(e_i)^{-1} = z$, then $g(e_i) z = z f(e_i)$ and $g(x) z = z f(x)$ for all $x \in C_L$.

If m is even, that is if C and $\mathfrak{C}$ are central simple algebras, by the Skolem–Noether theorem, there exists $s \in \mathfrak{C}_L$ such that $f(x) = s g(x) s^{-1}$ for all $x \in C_L$. Then we have $z s g(x) = z f(x) s = g(x) z s$ for all x. Therefore $zs \in L^*$ and $z = \lambda s^{-1}$ with $\lambda \in L^*$. Since s is an invertible element, z is invertible and $z^{-1} = \lambda^{-1} s$.

If m is odd, with a similar argument for C_0 and $\mathfrak{C}_0$, we obtain the result.

We fix elements $\{u_\sigma\} \in F(C_L)$ satisfying $\varphi_{C_L}(u_\sigma) = t(\sigma)$ with $u_1 = 1$. For $m_\sigma = g(u_\sigma) \in \mathfrak{C}_L$ we have the relation $a_{\sigma,\tau} = m_\sigma m_\tau m_{\sigma\tau}^{-1}$ and then $a_{\sigma,\tau} m_{\sigma\tau} = m_\sigma m_\tau$. Because $g^{-1} g^\sigma = t(\sigma)$, we obtain $g(x)^\sigma = m_\sigma g(x) m_\sigma^{-1}$ for all $x \in C$.

We have proved above that $f(e_i) z^{-1} = z^{-1} g(e_i)$ and, since $f(e_i) \in \mathfrak{C}$ (invariant by Γ), $f(e_i) z^{-\sigma} = z^{-\sigma} g(e_i)^\sigma$ and $f(e_i) z^{-\sigma} m_\sigma = z^{-\sigma} m_\sigma g(e_i)$. We put $b_\sigma := z z^{-\sigma} m_\sigma$. From the above relations it is clear to verify that b_σ commutes with $g(e_i)$ for all i. Thus, $b_\sigma \in Z(\mathfrak{C}_L)_0 = L$. Moreover, the elements b_σ satisfy the relation $b_\sigma b_\tau^\sigma = a_{\sigma,\tau} b_{\sigma\tau}$.

To compute $N(z)$, we consider $N(z z^{-\sigma}) = N(b_\sigma m_\sigma^{-1}) = b_\sigma^n N(m_\sigma^{-1}) = b_\sigma^n n_\sigma^{-1}$ and $N(z z^{-\sigma}) = \beta(z z^{-\sigma}) z z^{-\sigma} = \beta(z^\sigma)^{-1} \beta(z) z z^{-\sigma} = \beta(z^\sigma)^{-1} N(z) z^{-\sigma}$. Therefore $N(z) = b_\sigma^n n_\sigma^{-1} \beta(z^\sigma) z^\sigma = b_\sigma^n n_\sigma^{-1} N(z)^\sigma$.

If $\alpha \neq 0$ is a coordinate of $N(z)$ in a Γ-invariant basis of $\mathfrak{C}_0$, from the previous relations, we obtain $\alpha = b_\sigma^n n_\sigma^{-1} \alpha^\sigma$, and then $\alpha^{-1} = n_\sigma b_\sigma^{-n} \alpha^{-\sigma}$. If there exists an element $\eta \in L$ such that $\eta^{-\sigma} \eta = n_\sigma^{-1}$, the element $\gamma = \eta \alpha^{-1}$ satisfies $\gamma^\sigma \gamma^{-1} = b_\sigma^n$, and we obtain the theorem.□

Remark 8.3. If n is not a prime number, the solution given by γ may be not proper. But in any case, the set $\{L(\sqrt[n]{r\gamma}) \mid r \in K\}$ contains all proper and improper solutions.

According to this theorem, *the steps* to be followed in order *to obtain an element* γ *providing a solution* are:

(1) Find a representation of degree 0 such that $s_t = \varepsilon$ with the conditions of the theorem.
(2) Write down explicitly the isomorphism $g : C_L \to \mathfrak{C}_L$ over L such that $g^{-1}g^\sigma = t(\sigma)$ for each $\sigma \in \Gamma$.
(3) Determine an isomorphism $f : C \to \mathfrak{C}$ over K such that the element z be different from 0.
(4) Compute the expression of the element z in the basis of $\mathfrak{C}_L$.
(5) Compute the norm $N(z) \in \mathfrak{C}_L$ and consider a nonzero coordinate α of it.
(6) Find $\eta \in L$ such that $\eta^{-\sigma}\eta = n_\sigma^{-1}$ for all $\sigma \in \Gamma$.
(7) Compute the element $\gamma = \eta\alpha^{-1}$.
(In fact, here we compute γ^{-1} because of the difficulty to compute α^{-1} from α and the fact that the field $K(\sqrt[n]{\gamma^{-1}})$ is equal to $K(\sqrt[n]{\gamma})$.)

The main *difficulties* in achieving this process are, on the one hand:

(1) Finding the representation of degree 0.
(2) Expressing the solvability of the embedding problem in a way making possible the determination of the morphism f.

On the other hand, there are difficulties of a *computational nature*:

(3) Computing products and powers of elements in generalized Clifford algebras in order to compute the isomorphisms and the element z.
(4) Computing the norm of an element of degree 0 in a generalized Clifford algebra.

The last two difficulties are solved in the section below.

9. Computations in Generalized Clifford Algebras

In this section we carry out some computations in generalized Clifford algebras. We begin by fixing the basis in which we are going to work.

Definition 9.1. Let $C = C(e_1, e_2)$ be the generalized Clifford algebra generated by the elements e_1, e_2. The vector subspace of the elements of degree i of C is denoted by C_i . We consider the following basis of the subspaces C_i for $i = 0, \ldots, n-1$:

$$\omega^{c(k)-(i-1)k} e_1^{n+i-k} e_2^k, \qquad k = 0, \ldots, n-1$$

where $c(0) = 0$ and $c(k) = kx + \frac{k(k-1)}{2}$ for $k > 0$, where x is an arbitrary integer.

In the following theorem we compute the powers of elements in a generalized Clifford algebra which we use later to determine explicitly the isomorphism f over K.

THEOREM 9.2. *Let $C = C(e_1, e_2) = C(c, d)$ be the generalized Clifford algebra generated by the elements e_1, e_2 with $e_1^n = c$ and $e_2^n = d$.*

We put $D := (-1)^{n-1}d/c$. We consider the field $K(\sqrt[n]{D})$ and assume it is an extension of K of degree n. We fix $1, \sqrt[n]{D}, \ldots, \sqrt[n]{D^{n-1}}$ a basis of $K(\sqrt[n]{D})/K$. We denote by C_i the elements of degree i of C and, for every C_i we consider the basis defined above.
Let B be the element of C_1 with coordinates $x_0, \ldots, x_{n-1}$. We consider

$$\eta_0 = \sum_{k=0}^{n-1} x_k \sqrt[n]{D^k} \qquad \text{and} \qquad \eta_i = \sum_{k=0}^{n-1} x_k \omega^{ik} \sqrt[n]{D^k} \qquad \text{for } i = 1, \ldots, n-1$$

the conjugates of η_0. Then we have that

(a) If $k < n$, the matrix of the multiplication by B^k, that is, of $\cdot B^k : C_0 \longrightarrow C_k$ in the chosen basis is equal to the matrix of the multiplication by $c^k \eta_0 \cdots \eta_{k-1}$. Thus B^k has the same coordinates as $c^k \eta_0 \cdots \eta_{k-1}$, each of them written in the corresponding basis.
(b) If $n = k$, $B^n = c^{n+1} N_{K(\sqrt[n]{(-1)^{n-1}d/c})/K}(\eta_0)$.

PROOF. Let $[\cdot\eta_0]$ be the matrix of the multiplication by η_0 in the fixed basis. The multiplication by B on the right $(\cdot B)_{0,1} : C_0 \to C_1$ has the matrix $c[\cdot\eta_0]$.
The morphisms corresponding to the multiplication by η_i have the matrix

$$[\cdot\eta_i] = Diagonal[1, \omega, \omega^2, \ldots, \omega^{n-1}]^i \cdot [\cdot\eta_0] \cdot Diagonal[1, \omega, \omega^2, \ldots, \omega^{n-1}]^{-i}.$$

The multiplication by e_1 on the left $(e_1\cdot)_{i,i+1} : C_i \to C_{i+1}$ has the matrix

$$Diagonal[1, \omega, \ldots, \omega^{n-1}] \ \text{ if } i \neq n-1 \quad \text{and} \quad c \cdot Diagonal[1, \omega, \ldots, \omega^{n-1}] \ \text{ if } i = n-1,$$

and the multiplication by B

$$\begin{aligned}
&-[(\cdot B)_{1,2}] = [(e_1\cdot)_{1,2}][(\cdot B)_{0,1}][(e_1\cdot)_{0,1}]^{-1} = c[\cdot\eta_1],\\
&-[(\cdot B)_{i,i+1}] = c[\cdot\eta_i] \quad \text{if } i < n-1 \ \text{ and}\\
&-[(\cdot B)_{n-1,0}] = [(e_1\cdot)_{n-1,0}][(\cdot B)_{n-2,n-1}][(e_1\cdot)_{n-2,n-1}]^{-1} = c^2[\cdot\eta_{n-1}].
\end{aligned}$$

Considering the compositions

$$C_0 \xrightarrow{\cdot B} C_1 \xrightarrow{\cdot B} \cdots \xrightarrow{\cdot B} C_{k-1} \xrightarrow{\cdot B} C_k \qquad \text{and} \qquad C_0 \xrightarrow{\cdot B} C_1 \xrightarrow{\cdot B} \cdots \xrightarrow{\cdot B} C_{n-1} \xrightarrow{\cdot B} C_0,$$

we obtain the result.□

REMARK 9.3. Even if the field $K(\sqrt[n]{D})$ is not of degree n over K, the same result is true if we formally consider the elements $1, r, \ldots, r^{n-1}$ with $r^n = D$.

We have also computed the products $e_1 \cdot x, e_2 \cdot x$ and $x \cdot F$ for $x \in C_0$ or $x \in C_{n-1}$ and $F \in C_1$ or $F \in C_{n-1}$ in a generalized Clifford algebra. We use them in the package `n-Clifford`. To compute a coordinate of the norm of z we use the following result.

PROPOSITION 9.4. *Let $C = C(e_1, e_2) = C(c, d)$ be a generalized Clifford algebra. The subspace of the elements of degree i of C is denoted by C_i. Let $x = \sum_{i=0}^{n-1} x_i \omega^{c(i)+i} e_1^{n-i} e_2^i$ an element of degree 0.*

Then, the coefficient corresponding to $\omega^{c(k)+k}e_1^{n-k}e_2^k$ *of* $N(x) = \beta(x)x$ *is*

$$\sum_{j=0}^{n-k-1} x_{k+j}x_j^{n-1}\, c^{n-j-1}\, d^j + \sum_{j=n-k}^{n-1} x_{k+j-n}x_j^{n-1}(-1)^{n-1}c^{n-j}d^{j-1}.$$

In particular, for $k = 0$ *this coefficient is* $\alpha = \sum_{j=0}^{n-1} c^{(n-j-1)}d^j x_j^n$.

Proof. By definition of β (Definition 3.1), $\beta(x) = \sum_{i=0}^{n-1} x_i^{n-1}\omega^{-(c(i)+i)}e_2^{i(n-1)}e_1^{(n-i)(n-1)}$.

The expression $x_i x_j^{n-1}\omega^{c(i)-c(j)+i-j-ji+j^2}e_1^{n(n-j)-i+j}e_2^{i+j(n-1)}$ is the product of the summand j of $\beta(x)$ by the summand i of x. This summand corresponds to $k \equiv i - j \pmod n$. We study separately the case $i \geq j$, that is $i - k = j$, and the case $i < j$, that is $j = i - k + n$. The coefficient of $\omega^{c(k)+k}e_1^{n-k}e_2^k$ is

$$\sum_{j=0}^{n-k-1} x_{k+j}x_j^{n-1}c^{n-j-1}d^j + \sum_{j=n-k}^{n-1} x_{k+j-n}x_j^{n-1}c^{n-j}d^{j-1}(-1)^{n-1}.$$

In particular, if $k = 0$ this coefficient is $\sum_{j=1}^{n-1} x_j^n c^{n-j-1}d^j$.□

We develop the package `n-Clifford`, a set of programs in *Mathematica* designed to perform these computations.

10. Example 1: $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

Our first example is the Example 1 of Section 7. We take $L = K(\sqrt[n]{a})$ with $\Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$ generated by σ. The obstruction to the solvability of the embedding problem given by L/K, Γ and the exact sequence

$$(E) : 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{h} \mathbb{Z}/n\mathbb{Z} \to 0$$

is (a, ω). Therefore, the problem is solvable if and only if $(a, \omega) = 0$ if and only if ω is a norm in $K(\sqrt[n]{a})/K$ if and only if a is a norm in $K(\sqrt[n]{\omega})/K$. We recall that to compute this obstruction we have used a representation (C, t) of degree 0, where $C = C(e_1, e_2) = C(1, (-1)^{n-1}\omega)$, $t(\sigma) = \varphi(e_1^{n-1}e_2) \in O(C)$ and the twisted algebra is $\mathfrak{C} = C(v_1, v_2) = C(a, (-1)^{n-1}\omega a)$ where $v_1 = e_1 \otimes \sqrt[n]{a}$ and $v_2 = e_2 \otimes \sqrt[n]{a}$.

We suppose the problem is solvable. We want to find an element $\gamma \in L^*$ such that $M = L(\sqrt[n]{\gamma})$ is a solution.

10.1. The isomorphism $g : C_L \to \mathfrak{C}_L$ over L

Proposition 10.1. *The isomorphism*

$$g : C_L \to \mathfrak{C}_L, \qquad e_1 \mapsto \frac{1}{\sqrt[n]{a}}v_1, \qquad e_2 \mapsto \frac{1}{\sqrt[n]{a}}v_2$$

is a graded isomorphism that satisfies $g^{-1}g^\tau = t(\tau)$ *for each* $\tau \in \Gamma$.

PROOF. It is clear that g is a graded isomorphism of $C_L \to \mathfrak{C}_L$. To prove that $g^{-1}g^{\tau} = t(\tau)$ for all $\tau \in \Gamma$, as Γ is cyclic, it is only necessary to prove the relation for the generator σ. In this case

$$(g^{-1}g^{\sigma})(e_i) = g^{-1}\left(\frac{1}{\omega \sqrt[n]{a}}v_i\right) = \omega^{-1}g^{-1}\left(\frac{1}{\sqrt[n]{a}}v_i\right) = \omega^{-1}e_i = t(\sigma)(e_i). \square$$

10.2. THE ISOMORPHISM $f : C \to \mathfrak{C}$ OVER K

We are looking for an isomorphism $f : C \to \mathfrak{C}$ where

$$f(e_1) = \frac{1}{r_1}\sum_{k=0}^{n-1} A_k \omega^{c(k)} v_1^{n+1-k} v_2^k \qquad \text{and} \qquad f(e_2) = \frac{1}{r_1}\sum_{k=0}^{n-1} a_k \omega^{c(k)} v_1^{n+1-k} v_2^k$$

such that $f(e_1)^n = 1$, $f(e_2)^n = (-1)^{n-1}\omega$ and $f(e_1)f(e_2) = \omega f(e_2)f(e_1)$.

We apply Theorem 9.2 with $c = a$, $d = (-1)^{n-1}\omega a$ and then $D = \omega$. We fix the basis $1, \sqrt[n]{\omega}, \ldots, \sqrt[n]{\omega^{n-1}}$ of $K(\sqrt[n]{\omega})$ as K-vector space.

The power $f(e_1)^n = \frac{a^{n+1}}{r_1^n} N_{K(\sqrt[n]{\omega})/K}(\eta_0)$ where $\eta_0 = \sum_{k=0}^{n-1} A_k \sqrt[n]{\omega^k}$. Thus, $f(e_1)^n = 1$ if $N_{K(\sqrt[n]{\omega})/K}(\eta_0) = a^{n-1}$ and then $r_1 = a^2$. Since $(a, \omega) = 0$, there exists an element $y \in K(\sqrt[n]{\omega})$ with $N_{K(\sqrt[n]{\omega})/K}(y) = a$. The element $y^{n-1} \in K(\sqrt[n]{\omega})$ satisfies $N_{K(\sqrt[n]{\omega})/K}(y^{n-1}) = a^{n-1}$. Hence, if $y_0, \ldots, y_{n-1}$ are the coordinates of y^{n-1}, taking $A_k = y_k$ $(k = 0, \ldots, n-1)$ we obtain $N_{K(\sqrt[n]{\omega})/K}(\eta_0) = a^{n-1}$ and $f(e_1)^n = 1$. Similarly $f(e_2)^n = \frac{a^{n+1}}{r_1^n} N_{K(\sqrt[n]{\omega})/K}(\eta_0')$ where $\eta_0' = \sum_{k=0}^{n-1} a_k \sqrt[n]{\omega^k}$. Thus, $f(e_2)^n = (-1)^{n-1}\omega$ if $N_{K(\sqrt[n]{\omega})/K}(\eta_0') = (-1)^{n-1}\omega a^{n-1}$ and then $r_1 = a^2$. As $N_{K(\sqrt[n]{\omega})/K}(\sqrt[n]{\omega}) = (-1)^{n-1}\omega$, we can express

$$N_{K(\sqrt[n]{\omega})/K}(\eta_0') = (-1)^{n-1}\omega N_{K(\sqrt[n]{\omega})/K}(a_0\omega^{-1}\sqrt[n]{\omega^{n-1}} + a_1 + \cdots + a_{n-1}\sqrt[n]{\omega^{n-2}}).$$

Therefore we need elements a_k such that

$$N_{K(\sqrt[n]{\omega})/K}(a_0\omega^{-1}\sqrt[n]{\omega^{n-1}} + a_1 + \cdots + a_{n-1}\sqrt[n]{\omega^{n-2}}) = a^{n-1}.$$

Taking $a_0 = \omega y_{n-1}$ and $a_k = y_{k-1}, (k = 1, \ldots, n-1)$ where y_k are the coordinates of y^{n-1}, we obtain $f(e_2)^n = (-1)^{n-1}\omega$.

We only have to check the equality $f(e_1)f(e_2) = \omega f(e_2)f(e_1)$. The coefficient of $f(e_1)f(e_2)$ in $v_1^{n+2-k}v_2^k$ is the summand such that $i + j \equiv k \pmod n$, that is

$$a\left(\sum_{\substack{i+j\equiv k\\ i+j<n}} \omega^{c(i)+c(j)} A_i a_j \omega^{i(j-1)} + (-1)^{n-1}\omega \sum_{\substack{i+j\equiv k\\ i+j\geq n}} \omega^{c(i)+c(j)} A_i a_j \omega^{i(j-1)}\right) =: a \cdot Cf_1(k).$$

Similarly, the coefficient of $\omega f(e_2)f(e_1)$ in $v_1^{n+2-k}v_2^k$ is

$$\omega a\left(\sum_{\substack{i+j\equiv k\\ i+j<n}} \omega^{c(i)+c(j)} a_i A_j \omega^{i(j-1)} + (-1)^{n-1}\omega \sum_{\substack{i+j\equiv k\\ i+j\geq n}} \omega^{c(i)+c(j)} a_i A_j \omega^{i(j-1)}\right) =: a\omega \cdot Cf_2(k).$$

Thus, $f(e_1)f(e_2) = \omega f(e_2)f(e_1)$ if and only if for any $k = 0, \dots, n-1$

$$Cf_1(k) = \omega Cf_2(k) \Leftrightarrow (*) := \sum_{i=0}^{k} \omega^{c(i)+c(k-i)} A_i a_{k-i} \omega^{i(k-i)} (\omega^{-i} - \omega^{1-k+i})$$
$$+\omega \sum_{i=k+1}^{n-1} \omega^{c(i)+c(k-i)} A_i a_{k-i} \omega^{i(k-i)} (\omega^{-i} - \omega^{1-k+i}) = 0.$$

We fix any k and prove that $(*) = 0$. A summand of the previous expression is 0 for an i such that $\omega^{-i} - \omega^{1-k+i} = 0$ if and only if $2i \equiv k-1 \pmod n$.

If n is odd there is only a summand that is 0, the one corresponding to $i \equiv 2^{-1}(k-1) \pmod n$, that is the one corresponding to $i \equiv \frac{n+1}{2}(k-1) \pmod n$.

If n is even we have two cases. If k is even, $2i \equiv k-1 \pmod n$ has no solution. If k is odd, $i \equiv \frac{k-1}{2} \pmod{\frac{n}{2}}$ has two solutions: $i = \frac{k-1}{2}$ and $i = \frac{k-1+n}{2}$. The first one is $i \le k-1$ and the second one is $n-1 \ge i \ge k$.

We group in pairs the summand different from 0 of $(*)$, the summands $A_i a_{k-i} = y_i y_{k-i-1}$ with $A_{k-i-1} a_{i+1}$. If $i \equiv k-i-1 \pmod n$, these summands coincide. This occurs if $2i \equiv k-1 \pmod n$ and in these cases the summands are 0. We remark that if $0 \le i \le k-1$, then $0 \le k-i-1 \le k-1$. If $k+1 \le i \le n-2$, then $k+1 \le n+k-i-1 \le n-2$ and if $i = k$ then $k-i-1 = -1 \equiv n-1 \pmod n$. Thus, if $i \ne k$ are both in the same summation of $(*)$ and their sum is S if $i \le k-1$ and $(-1)^{n-1}\omega S$ if $n-1 > i > k$, where

$$S = y_i y_{k-i-1} \omega^{i(k-i)} (\omega^{-i} - \omega^{1-k+i})(\omega^{c(i)+c(k-i)} - \omega^{c(k-i-1)+c(i+1)+k-2i-1}) = 0$$

because $c(i) + c(k-i) \equiv c(k-i-1) + c(i+1) + k - 2i - 1 \pmod n$.

If $i = k$, adding the summand i with the summand $n-1$ we obtain

$$y_k y_{n-1} (\omega^{-k} - \omega)(\omega^{c(k)+1} - \omega^{c(n-1)+c(k-n+1)-k}) = 0$$

because $c(k) + 1 \equiv c(n-1) + c(k-n+1) - k \pmod n$.

Hence we have proved that $f(e_1)f(e_2) = \omega f(e_2)f(e_1)$ and the next theorem.

Theorem 10.2. *We have the following graded isomorphism over K:*

$$f : C \to \mathfrak{C}$$
$$e_1 \mapsto \frac{1}{a^2}(y_0 v_1^{n+1} + \cdots + \omega^{c(k)} y_k v_1^{n+1-k} v_2^k + \cdots + \omega^{c(n-1)} y_{n-1} v_1^2 v_2^{n-1})$$
$$e_2 \mapsto \frac{1}{a^2}(\omega y_{n-1} v_1^{n+1} + \cdots + \omega^{c(k)} y_{k-1} v_1^{n+1-k} v_2^k + \cdots + \omega^{c(n-1)} y_{n-2} v_1^2 v_2^{n-1}),$$

where $y_0 + y_1 \sqrt[n]{\omega} + \cdots + y_{n-1} \sqrt[n]{\omega^{n-1}}$ is an element of $K(\sqrt[n]{\omega})$ with norm a^{n-1}.

We can modify $f(e_1)$ and/or $f(e_2)$ by multiplying them by an adequate nth root of the unity if it is necessary to assure $z \ne 0$.

10.3. THE ELEMENT z

From the isomorphisms f and g, we can compute the element $z \in \mathfrak{C}_L$, homogeneous of degree 0. In this case the expression of z is

$$z = \sum_{\epsilon_i \in \{0,\dots,n-1\}} g(e_1)^{\epsilon_1} g(e_2)^{\epsilon_2} f(e_2)^{-\epsilon_2} f(e_1)^{-\epsilon_1}.$$

We consider in $\mathfrak{C}_i$ the basis defined in 9.1 and in $K(\sqrt[n]{\omega})$ the basis 1, $\sqrt[n]{\omega}, \ldots, \sqrt[n]{\omega^{n-1}}$. We have programmed a function to compute z. For this, it is necessary to compute the expression of $f(e_1)^{-1}$ and $f(e_2)^{-1}$.

Proposition 10.3. *(a)* $f(e_1)^{-1} = f(e_1)^{n-1}\frac{1}{a^{n-1}}\sum_{k=0}^{n-1} B_k(\omega^{c(k)+2k} v_1^{n-1-k} v_2^k)$ *where* $\eta_0 = \sum_{k=0}^{n-1} y_k \sqrt[n]{\omega^k} \in K(\sqrt[n]{\omega})$, $\eta_i = \sum_{k=0}^{n-1} y_k \omega^{ik} \sqrt[n]{\omega^k}$ *the conjugates of* η_0 *and we denote by* $(B_0, \ldots, B_{n-1})$ *the coordinates of* $\eta_0 \cdots \eta_{n-2} \in K(\sqrt[n]{\omega})$.

(b) $f(e_2)^{-1} = (-1)^{n-1}\omega^{-1} f(e_2)^{n-1}$. *Let* $\eta'_0 = \omega y_{n-1} + \sum_{k=1}^{n-1} y_{k-1} \sqrt[n]{\omega^k} \in K(\sqrt[n]{\omega})$ *and* η'_i *its conjugates. We denote by* $(D_0, \ldots, D_{n-1})$ *the coordinates of* $\eta'_0 \cdots \eta'_{n-2} \in K(\sqrt[n]{\omega})$. *Then* $f(e_2)^{n-1} = \frac{1}{a^{n-1}}\sum_{k=0}^{n-1} D_k(\omega^{c(k)+2k} v_1^{n-1-k} v_2^k)$.

This proposition is a consequence of Theorem 9.2.

For instance, for $n = 3$, the coordinates of the element z in the fixed basis are

$$z = \frac{1}{a^{4/3}}\{\, y_0^2 + a^{2/3} y_0 + a^{4/3} - \omega\, y_1\, y_2 \,, \omega\, y_2^2 - (-\omega^2 a^{2/3} + y_0)\, y_1 \,, y_1^2 + (a^{2/3}\,\omega - y_0)\, y_2 \,\}$$

where $y_0 + y_1 \sqrt[3]{\omega} + y_2 \sqrt[3]{\omega^2}$ is the element $y^2 \in K(\sqrt[3]{\omega})$ with norm a^2.

The expression for $n = 4$ can be found in Vela (1998a).

10.4. the element γ

Below, we compute theoretically the element $\gamma \in L^*$ such that $M = L(\sqrt[n]{\gamma})$. We know that $\gamma = \eta\alpha^{-1}$, where α is a coordinate of $N(z)$ and η satisfies $\eta^{-\sigma}\eta^{-1} = N(m_\sigma)^{-1}$ for each $\sigma \in \Gamma$ where $m_\sigma = g(u_\sigma)$. In Proposition 9.4 we compute the coordinates of the norm of an element of degree 0. In particular, if $(x_0, \ldots, x_{n-1})$ are the coordinates of the element $z \in \mathfrak{C}_0$, we can take as α the element $\alpha = a^{n-1}\sum_{j=0}^{n-1}\omega^j(-1)^{j(n-1)}x_j^n$ if it is different from 0.

Theorem 10.4. *Let* $(x_0, \ldots, x_{n-1})$ *be the coordinates of the element* $z \in \mathfrak{C}_0$. *Let*

$$\alpha = a^{n-1}\sum_{j=0}^{n-1}\omega^j(-1)^{j(n-1)}x_j^n \qquad \text{and} \qquad \eta = \begin{cases} \sqrt[n]{a} & \text{if } n \text{ is odd and} \\ \sqrt[n]{a^{(n/2)+1}} & \text{if } n \text{ is even.} \end{cases}$$

Then, the element $\gamma = \eta\alpha^{-1}$ *is such that* $M = L(\sqrt[n]{\gamma})$ *is a solution to the given embedding problem.*

Proof. To prove this result it is only necessary to prove that $\eta^{-\sigma}\eta = N(m_\sigma)^{-1}$ for $m_\sigma = g(u_\sigma)$. From the representation t, we have $u_\sigma = e_1^{n-1}e_2$ because $t(\sigma) = \varphi_C(e_1^{n-1}e_2)$. Then $m_\sigma = g(u_\sigma) = \frac{1}{a}v_1^{n-1}v_2$ and $N(m_\sigma) = (-1)^{n-1}\omega$.

It is straightforward to check that $\eta^{-\sigma}\eta = n_\sigma^{-1} = (-1)^{n-1}\omega^{-1}$.□

Table 1. Example 1. Results for $n = 3$.

a	γ_a
3	$9 + 6\ 3^{1/3} + 4\ 3^{2/3}$
17	$782\ \sqrt[3]{17}\omega - 2023\ \omega + 304\ \sqrt[3]{17^2} - 2023$
19	$1235\ \sqrt[3]{19}\omega - 3249\ \omega + 505\ \sqrt[3]{19^2} - 3249$
37	$(339576\omega - 120388)\ \sqrt[3]{37^2}\ + (1483108\ \omega + 421726)\sqrt[3]{17} + 8964212\ \omega\ - 1967253$
57	$(50139\omega - 16475)\sqrt[3]{57^2} + (-39729\omega + 172197)\sqrt[3]{57} - 708282\ \omega - 500346$
73	$(235565469\omega - 181757491)\sqrt[3]{73^2} + (2488516564\omega + 336612271)\sqrt[3]{73}$ $+5864042258\omega + 5264391204$
109	$(234080028\omega - 119604320)\sqrt[3]{109^2} + (-1755609372\omega - 899497430)\sqrt[3]{109}$ $+731822076\omega + 5778882757$
$\omega + 2$	$(\omega + 2)\sqrt[3]{(\omega + 2)^2} + (\omega - 1)\sqrt[3]{\omega + 2} + 3\omega + 1$
$\omega - 1$	$(\omega + 2)\sqrt[3]{(\omega - 1)^2} + (-\omega + 1)\sqrt[3]{\omega - 1} + 3\omega + 2$

Table 2. Example 1. Results for $n = 4$.

a	γ_a
7	$-27440\imath - 20629\ \sqrt[4]{7} + 14364\imath\sqrt{7} + 7831\ \sqrt[4]{7^3}$
-7	$-236327i + 106477(1 + i)\sqrt[4]{-7} + 941192(1 + i)\sqrt[4]{(-7)^3}$ $-194404i\sqrt{7} + 932871(1 - i)\sqrt[4]{-7^3}$
8	$10\sqrt[4]{2} + 7\sqrt[4]{2^3}$
-14	$4733400(1 - i) + 6339375\ \sqrt[4]{-14} + 5647152\ i\sqrt[4]{(-14)^3}$ $+6319250(1 - i)\sqrt{14} + 3646152\ \sqrt[4]{-14^3}$
17	$-3463665\imath - 1953351\ \sqrt[4]{17} + 836893\imath\ \sqrt{17} + 473487\ \sqrt[4]{17^3}$
18	$20808 + 108000\imath + (543024 + 1921860\imath)\sqrt{2}+$ $(21180 + 198097\imath)\ \sqrt[4]{2}\sqrt{3} + (1595430 + 2601571\ i)\sqrt[4]{2^3}\sqrt{3}$
23	$15330420 + 7265815\ \sqrt[4]{23} + 3207764\ \sqrt{23} + 1625651\ \sqrt[4]{23^3}$
31	$59403254(-1 + i) + 74978181\ \sqrt[4]{31} - 9523014(1 + i)\sqrt{31} + 5043057\ i\sqrt[4]{31^3}$
-31	$-165906079\imath + 65357610\ \sqrt[4]{-31} - 9374493\sqrt{31} + 6102966\sqrt[4]{-31^3}$
-34	$-142614564 + 366804580\imath + (54073345 + 70851240\imath)\sqrt[4]{-34}$ $+(27299930 + 151010551\imath)\sqrt[4]{-34^3} + (650760 + 210361401\imath)\sqrt{34}$
-97	$-226470264041691216(1 + i) - 1297253101634549215\ \sqrt[4]{-97} + (-461133301928694401\imath$ $+321461357878865679)\sqrt[4]{-97^3} - 333371425206635056(-1 + i)\sqrt{97}$

For instance, we show here the general formula for $n = 3$:

$$\begin{aligned}\gamma = \frac{1}{a^{7/3}}(3\ (\omega^2\ y_1^6 + a^{2/3}\ (\omega + 2)\ y_2\ y_1^4 + (y_2^3 - 3\ a^2\ \omega)\ y_1^3 - a^{2/3}\ (\omega + 2)\ y_2^4\ y_1 \\ +\omega\ y_2^6 + 3\ a^4 + 3\ a^2\ y_2^3 + 3\ a^2\ \omega\ y_2^3 + y_0^2\ (-a^{2/3}\ (\omega - 1)\ y_1^3 + 3\ \omega^2\ y_2^2\ y_1^2 \\ +3\ a^{4/3}\ \omega\ y_2\ y_1 + a^{2/3}\ (\omega\ y_2^3 + 2\ y_2^3 + 3\ a^2)) + y_0\ (-3\ \omega^2\ y_2\ y_1^4 \\ -a^{4/3}\ (\omega - 1)\ y_1^3 + 6\ a^{2/3}\ \omega^2\ y_2^2\ y_1^2 + (6\ a^2\ \omega\ y_2 - 3\ y_2^4)\ y_1 + a^{4/3}\ (\omega\ y_2^3 \\ +2\ y_2^3 + 3\ a^2))))\end{aligned}$$

where $y_0 + y_1\sqrt[3]{\omega} + y_2\sqrt[3]{\omega^2}$ is the element $y^2 \in K(\sqrt[3]{\omega})$ with norm a^2.

The general formula for $n = 4$ can be consulted in Vela (1998a). The values of γ for some specific values of a and $n = 3, 4$ are given in Tables 1 and 2. Similar tables for $n = 5, 6$ can be found in Vela (1998a).

11. Example 2: $0 \to \mathbb{Z}/n\mathbb{Z} \to E_1 \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to 0$

In Theorem 7.2 we study the embedding problem given by $L = K(\sqrt[n]{a_1}, \sqrt[n]{a_2})/K$ with $\Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ generated by h_1, h_2 such that $h_i(\sqrt[n]{a_j}) = \omega^{\delta_{ij}} \sqrt[n]{a_j}$ and the exact sequence

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} E_1 \xrightarrow{h} \Gamma \to 0$$

where E_1 is the group

$$E_1 = \langle \sigma, \tau \mid \sigma \text{ has order } n, \tau \text{ has order } n^2, \sigma\tau = \tau^{n+1}\sigma \rangle.$$

The obstruction to the solvability is the symbol $(\omega^{-1}a_1, a_2)$. We computed this obstruction with a representation which does not have degree 0. To apply Theorem 8.2 we need a representation of degree 0 such that $\varepsilon = s_t$.

We consider the representation $t : \Gamma \to O(C), h_1 \mapsto \varphi(e_0^{-1}e_1), h_2 \mapsto \varphi(e_0^{-1}e_2)$, where $C = C_t = C(e_0, e_1, e_2) = C((-1)^{n-1}, 1, \omega)$ with $A_t = \{x = \lambda e_0^{\epsilon_0} e_1^{\epsilon_1} e_2^{\epsilon_2}\} \subset F(C_t)$ and the admissible norm $\mathcal{N}(e_i) = 1 (i = 0, 1, 2)$. The twisted algebra by t is $\mathfrak{C} = C(v_0, v_1, v_2) = C((-1)^{n-1}a_1a_2, a_1a_2^2, \omega a_2)$ where $v_0 = e_0 \otimes \sqrt[n]{a_1a_2}$, $v_1 = e_1 \otimes \sqrt[n]{a_1a_2^2}$ and $v_2 = e_2 \otimes \sqrt[n]{a_2}$ are fixed by the action of Γ. Clearly this representation is of degree 0 and, as $(e_0^{-1}e_i)^k = \omega^{\frac{k(k-1)}{2}} e_0^{-k} e_i^k$ $i = 1, 2$, it is easy to check that $\varepsilon = s_t$.

We suppose the problem is solvable. We want to find an element $\gamma \in L^*$ such that $M = L(\sqrt[n]{\gamma})$ is a solution. To simplify the notation we put $a = a_1a_2$.

11.1. THE ISOMORPHISM $g : C_L \to \mathfrak{C}_L$ OVER L

PROPOSITION 11.1. *The isomorphism*

$$g : C_L \to \mathfrak{C}_L, \qquad e_0 \mapsto \frac{1}{\sqrt[n]{a}} v_0, \qquad e_1 \mapsto \frac{1}{\sqrt[n]{a_1a_2^2}} v_1, \qquad e_2 \mapsto \frac{1}{\sqrt[n]{a_2}} v_2$$

is a graded isomorphism which satisfies $g^{-1}g^\tau = t(\tau)$ for each $\tau \in \Gamma$.

The proof is similar to that in Proposition 10.1.

We express the isomorphism g as the composition of the two isomorphisms g_1 and g_2

$$C_L = C((-1)^{n-1}, 1, \omega)_L \overset{g_1}{\simeq} C'_L = C((-1)^{n-1}a, a, \omega)_L \overset{g_2}{\simeq} C((-1)^{n-1}a, aa_2, \omega a_2)_L = \mathfrak{C}_L$$

given by $g_1(e_0) = \frac{1}{\sqrt[n]{a}} u_0$, $g_1(e_1) = \frac{1}{\sqrt[n]{a}} u_1$, $g_1(e_2) = u_2$, where the elements u_0, u_1 and u_2 generate $C((-1)^{n-1}a, a, \omega)$ and $g_2(u_0) = v_0$, $g_2(u_1) = \frac{1}{\sqrt[n]{a_2}} v_1$, $g_2(u_2) = \frac{1}{\sqrt[n]{a_2}} v_2$. Clearly $g = g_2 \circ g_1$.

11.2. THE ISOMORPHISM $f : C \to \mathfrak{C}$ OVER K

We express this isomorphism over K in terms of two easier isomorphisms between generalized Clifford algebras generated by two elements. We consider the isomorphisms

$$C = C((-1)^{n-1}, 1, \omega) \overset{f_1}{\simeq} C'_L = C((-1)^{n-1}a, a, \omega) \overset{f_2}{\simeq} C((-1)^{n-1}a, aa_2, \omega a_2) = \mathfrak{C}$$

where the first one is obtained from the isomorphism $C((-1)^{n-1}, 1) \simeq C((-1)^{n-1}a, a)$, valid for any value of $a \in K$, and the second one from the isomorphism $C(a_1a_2, \omega) \simeq C(a_1a_2^2, \omega a_2)$ whose existence is given by the solvability of the embedding problem corresponding to a_1, a_2.

THE ISOMORPHISM $f_1 : C((-1)^{n-1}, 1) \simeq C((-1)^{n-1}a, a)$

We put $f_1(e_0) = \frac{1}{r_0}\sum_{k=0}^{n-1} A_k \omega^{c(k)} u_0^{n+1-k} u_1^k$ and $f_1(e_1) = \frac{1}{r_1}\sum_{k=0}^{n-1} \nu_k \omega^{c(k)} u_0^{n+1-k} u_1^k$ and we are looking for the values r_0, r_1, A_k and ν_k such that $f_1(e_0)^n = (-1)^{n-1}$, $f_1(e_1)^n = 1$ and $f_1(e_0)f_1(e_1) = \omega f_1(e_1)f_1(e_0)$. We apply formally Theorem 9.2. The power $f_1(e_0)^n = \frac{(-1)^{n-1}a^{n+1}}{r_0^n}\eta_0 \cdots \eta_{n-1}$ where $\eta_0 = \sum_{k=0}^{n-1} A_k\, \omega^k$ and $\eta_i = \sum_{k=0}^{n-1} \omega^{ik} A_k\, \omega^k$, $i = 1, \ldots, n-1$. Thus, $f_1(e_0)^n = (-1)^{n-1}$ if $a^{n+1}\eta_0 \cdots \eta_{n-1}$ is an nth power and then, $r_0 = a^2$. It is sufficient to find values of A_k such that $\eta_0 \cdots \eta_{n-1} = a^{n-1}$.

REMARK 11.2. We have the equality $(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = n$.

LEMMA 11.3. *Let* $A_0 = \frac{a^{n-1}}{n^2} + 1, A_1 = (\frac{a^{n-1}}{n^2} - 1)\omega^{-1}$ *and* $A_k = \frac{a^{n-1}}{n^2}\ \omega^{-k}$ *for* $k \geq 2$. *With these expressions,* $\eta_0 = \frac{a^{n-1}}{n}$ *and* $\eta_i = 1 - \omega^i$ *for* $i > 0$. *Hence* $\eta_0 \cdots \eta_{n-1} = a^{n-1}$.

Analogously we look for ν_k such that $f_1(e_1)^n = 1$. We consider the elements $\eta'_0 = \sum_{k=0}^{n-1} \nu_k \omega^k$ and $\eta'_i = \sum_{k=0}^{n-1} \nu_k \omega^{k(i+1)}$, $i = 1, \ldots, n-1$.

LEMMA 11.4. *By defining* $\nu_0 = A_{n-1}$ *and* $\nu_k = \omega^{c(k)} A_{k-1}$ *if* $k \geq 1$, *we get*

$$\eta'_0 \cdots \eta'_{n-1} = (-1)^{n-1} a^{n-1}.$$

In this case $f_1(e_1)^n = \frac{(-1)^{n-1}a^{n+1}}{r_1^n}\eta'_0 \cdots \eta'_{n-1} = \frac{1}{r_1^n}a^{2n}$ and, taking $r_1 = a^2$, the power $f_1(e_1)^n = 1$.

The equality $f_1(e_0)f_1(e_1) = \omega f_1(e_1)f_1(e_0)$ is proved with a similar computation to that in Example 1.

THE ISOMORPHISM $f_2 : C(a_1a_2, \omega) \simeq C(a_1a_2^2, \omega a_2)$

The existence of this isomorphism is given by the solvability of the embedding problem. We apply Theorem 9.2 for $c = a_1a_2^2$, $d = \omega a_2$ and $D = (-1)^{n-1}d/c = (-1)^{n-1}\omega a^{-1}$ (because $a = a_1a_2$). In this case $(\omega^{-1}a_1, a_2) = 0$ and, by properties of Galois symbols, $(\omega a_1^{-1}a_2^{-2}, D) = (\omega^{-1}a_1, a_2) = 0$. Working like the previous example we get the following theorem.

THEOREM 11.5. *The graded isomorphism over* K *is composition of the isomorphisms*

$$f_1 : C((-1)^{n-1}, 1, \omega) \to C((-1)^{n-1}a, a, \omega)$$

$$e_0 \mapsto \frac{1}{a^2}\sum_{k=0}^{n-1} A_k \omega^{c(k)} u_0^{n+1-k} u_1^k, \quad e_1 \mapsto \frac{1}{a^2}\sum_{k=0}^{n-1} \nu_k \omega^{c(k)} u_0^{n+1-k} u_1^k, \quad e_2 \mapsto u_2$$

with

$$A_0 = \frac{a^{n-1}}{n^2} + 1,\ A_1 = \left(\frac{a^{n-1}}{n^2} - 1\right)\omega^{-1}, A_k = \frac{a^{n-1}}{n^2}\omega^{-k} \text{ if } k \geq 2 \text{ and}$$

$$\nu_0 = \frac{a^{n-1}}{n^2}\omega,\ \nu_1 = \frac{a^{n-1}}{n^2}+1,\ \nu_2 = \left(\frac{a^{n-1}}{n^2}-1\right)\omega^{-1}, \nu_k = \frac{a^{n-1}}{n^2}\omega^{-k+1} \text{ if } k \geq 3$$

and

$$f_2 : C((-1)^{n-1}a, a, \omega) \to C((-1)^{n-1}a, aa_2, \omega a_2)$$

$$u_0 \mapsto v_0, \quad u_1 \mapsto \frac{1}{aa_2}\left(\sum_{k=0}^{n-2} y_{k+1}\ \omega^{c(k)} v_1^{n+1-k} v_2^k + y_0 D^{-1}\omega^{c(n-1)} v_1^2 v_2^{n-1}\right),$$

$$u_2 \mapsto \frac{1}{aa_2}\sum_{k=0}^{n-1} y_k\ \omega^{c(k)} v_1^{n+1-k} v_2^k,$$

where $D = (-1)^{n-1}\omega a^{-1}$ *and* $y_0 + y_1\sqrt[n]{D} + \cdots + y_{n-1}\sqrt[n]{D^{n-1}}$ *is an element with norm* $\omega a_1^{-1} a_2^{-2}$.

We can modify $f(e_i)$ multiplying it by an adequate nth root of the unity if it is necessary to assure $z \neq 0$.

11.3. the element z

From the isomorphisms f_1, g_1 and f_2, g_2, we can compute elements $z_1 \in C'_L$ and $z_2 \in \mathfrak{C}_L$ such that $g_i(x) = z_i f_i(x) z_i^{-1}$ for $i = 1, 2$ and x any element of the corresponding Clifford algebra. The element $z = g_2(z_1)\, z_2 \in \mathfrak{C}_L$ satisfies $g(x) = zf(x)z^{-1}$ for all $x \in C_L$.

We now compute the elements z_1 and z_2. As $f_1(e_2) = g_1(e_2)$, we can express

$$z_1 = \sum_{\epsilon_i \in \{0,\ldots,n-1\}} g_1(e_0)^{\epsilon_0} g_1(e_1)^{\epsilon_1} f_1(e_1)^{-\epsilon_1} f_1(e_0)^{-\epsilon_0},$$

and, as $f_2(e_0) = g_2(e_0)$,

$$z_2 = \sum_{\epsilon_i \in \{0,\ldots,n-1\}} g_2(u_1)^{\epsilon_1} g_2(u_2)^{\epsilon_2} f_2(u_2)^{-\epsilon_2} f_2(u_1)^{-\epsilon_1}.$$

We compute similarly to the previous example using Theorem 9.2, the expressions of $f_1(e_0)^{-1}$, $f_1(e_1)^{-1}$, $f_2(u_1)^{-1}$ and $f_2(u_2)^{-1}$ and the elements z_1 and z_2.

For instance, for $n = 3$ the coordinates of the elements z_1 and z_2 in the fixed basis are

$$\begin{aligned} z_1 = \{&a^{8/3} + 3\ a^2\ + 9\ a^{4/3} + 9\ a^{2/3} + 9, \omega\ a^{8/3} - 9\ \omega\ a^{2/3} + 9\ \omega^2,\\ &\omega^2\ a^{8/3} - 3\ \omega\ a^2 + 9\ \omega\},\\ z_2 = \{&-a_2^{2/3}\ y_1^2 - \sqrt[3]{a_2}\ y_1 + a_2^{2/3}\ y_0\ y_2 - 1,\\ &\sqrt[3]{a_2}\ (\omega\ a_1\ a_2^{4/3}\ y_0^2 + a_1\ a_2^{4/3}\ y_0^2 + \omega\ y_2 + \sqrt[3]{a_2}\ y_1\ y_2 + y_2),\\ &-a_2^{2/3}(y_2^2 + a_1\ a_2^{2/3}\ y_0 + \omega\ a_1\ a_2\ y_0\ y_1 + a_1\ a_2\ y_0\ y_1)\} \end{aligned}$$

where $y_0 + y_1\sqrt[3]{\omega a^{-1}} + y_2\sqrt[3]{(\omega a^{-1})^2} = y \in K(\sqrt[3]{\omega a^{-1}})$ such that $N_{K(\sqrt[3]{\omega a^{-1}})/K}(y) = \omega a_1^{-1} a_2^{-2}$.

The general expression of the elements z_1 and z_2 for $n = 4$ can be found in Vela (1998a).

11.4. THE ELEMENT γ

We look now for an element $\gamma \in L$ such that $M = L(\sqrt[n]{\gamma})$ is a solution to the embedding problem. We compute $\gamma = \gamma_1 \cdot \gamma_2$ where $\gamma_1 \in L$ is the element corresponding to z_1 and $\gamma_2 \in L$ is the element corresponding to z_2. We know (see the proof of Theorem 8.2) that the element $b_\sigma := zz^{-\sigma} g(u_\sigma) \in L$ satisfies $b_\sigma b_\tau^\sigma = a_{\sigma,\tau} b_{\sigma\tau}$ and the element γ satisfies the relation $\gamma^\sigma \gamma^{-1} = b_\sigma^n$.

We consider the representations

$$\begin{array}{ll} T : \Gamma \to O(C), & h_1 \mapsto \varphi_C(e_0^{-1} e_1), \quad h_2 \mapsto \varphi_C(e_0^{-1} e_1), \\ T' : \Gamma \to O(C'), & h_1 \mapsto 1, \quad h_2 \mapsto \varphi_{C'}(\sqrt[n]{a} u_1^{-1} u_2). \end{array}$$

It is clear that $g_1^{-1} g_1^\sigma = T(\sigma)$ and $g_2^{-1} g_2^\sigma = T'(\sigma)$. We consider for T the elements

$$M_{h_1} = M_{h_2} = g_1(e_0^{-1} e_1) = u_0^{-1} u_1, \qquad N_{h_1} = N_{h_2} = N(M_{h_1}) = (-1)^{n-1},$$

$$M_\sigma = g_1(u_\sigma), \qquad N_\sigma = N(M_\sigma) \qquad \text{and} \qquad B_\sigma = z_1 z_1^{-\sigma} M_\sigma \in L, \ \forall\, \sigma \in \Gamma.$$

They satisfy $N(z_1) = B_\sigma^n N_\sigma^{-1} N(z_1)^\sigma$. Let α_1 be a coordinate of $N(z_1)$. The element $\eta_1 = 1$ if n is odd, $\eta_1 = \sqrt[n]{(a_1 a_2)^{n/2}}$ if n is even, satisfies $\eta_1^{-\sigma} \eta_1 = n_\sigma^{-1}$. Then, the element $\gamma_1 = \eta_1 \alpha_1^{-1}$ satisfies the relation $\gamma_1^\sigma = B_\sigma^n \gamma_1$ for each $\sigma \in \Gamma$.

Similarly, we consider for T' the elements

$$M'_{h_1} = g_2(1) = 1, \quad M'_{h_2} = g_2(\sqrt[n]{a} u_1^{-1} u_2) = \sqrt[n]{a} v_1^{-1} v_2, \quad M'_\sigma = g_2(u'_\sigma), N'_\sigma = N(M_\sigma)$$

and $B'_\sigma = z_2 z_2^{-\sigma} M'_\sigma \in L$, for all $\sigma \in \Gamma$. They satisfy $N(z_2) = B'^n_\sigma N'^{-1}_\sigma N(z_2)^\sigma$. In this case the element $\eta_2 = \sqrt[n]{a_2}$ satisfies $\eta_2^{-\sigma} \eta_2 = N'^{-1}_\sigma$ and if α_2 is a coordinate of $N(z_2)$, the element $\gamma_2 = \eta_2 \alpha_2^{-1}$ satisfies the relation $\gamma_2^\sigma = b'^n_\sigma \gamma_2$ for each $\sigma \in \Gamma$.

It should be noted that for the representation t, the elements $m_\sigma = g(u_\sigma) = g_2(M_\sigma M'_\sigma)$. Moreover, we have the following relation.

LEMMA 11.6. *$b_\sigma = B_\sigma B'_\sigma \nu_\sigma$ where $\nu_\sigma \in \mu_n$. Therefore $b_\sigma^n = B_\sigma^n B'^n_\sigma$.*

Then, the element $\gamma = \gamma_1 \gamma_2$ satisfies $\gamma^\sigma = B_\sigma^n \gamma$ and $M(\sqrt[n]{\gamma})$ is a solution to the embedding problem.

PROOF. We have

$$\begin{array}{l} M_\sigma = \rho_\sigma u_0^{-r_\sigma}\ u_1^{r_\sigma} \Rightarrow g_2(M_\sigma) = \rho_\sigma \sqrt[n]{a_2^{r_\sigma}} v_0^{-r_\sigma}\ v_1^{r_\sigma}, \quad \text{and} \\ M'_\sigma = \rho'_\sigma \sqrt[n]{a^{s_\sigma}} v_1^{-s_\sigma}\ v_2^{s_\sigma} \text{ where } \rho_\sigma \rho'_\sigma \in \mu_n, r_\sigma, s_\sigma \in \mathbb{N}. \end{array}$$

Then $m_\sigma = g_2(M_\sigma) M'_\sigma = \sqrt[n]{a_1^{s_\sigma} a_2^{s_\sigma - r_\sigma}} \rho_\sigma \rho'_\sigma v_0^{-r_\sigma}\ v_1^{r_\sigma - s_\sigma}\ v_2^{s_\sigma}$. By computing

$$b_\sigma = z\ z^{-\sigma}\ m_\sigma = B'_\sigma g_2(z_1)\ v_2^{-s_\sigma}\ v_1^{s_\sigma}\ g_2(z_1)^{-\sigma} \sqrt[n]{a_2^{-r_\sigma}} \rho_\sigma v_0^{-r_\sigma}\ v_1^{r_\sigma - s_\sigma}\ v_2^{s_\sigma}$$

and

$$\frac{b_\sigma \sqrt[n]{a_2^{r_\sigma}}}{B'_\sigma \rho_\sigma} v_1^{s_\sigma - r_\sigma}\ v_0^{r_\sigma}\ g_2(z_1)^\sigma = g_2(z_1)\ v_1^{s_\sigma},$$

since $z_1 = \sum_{k=0}^{n-1} x_k \omega^{c(k)+k} u_0^{n-k} u_1^k$, then

$$v_1^{s_\sigma - r_\sigma}\ v_0^{r_\sigma}\ g_2(z_1)^\sigma = \sum_{k=0}^{n-1} \sqrt[n]{a_2^{-k}} x_k^\sigma \tau_{\sigma,k} v_0^{n-k+r_\sigma}\ v_1^{k+s_\sigma - r_\sigma} \text{ where } \tau_{\sigma,k} \in \mu_n$$

and $g_2(z_1)\ v_1^{s_\sigma} = \sum_{k=0}^{n-1} \sqrt[n]{a_2^{-k}} x_k \omega^{c(k)+k} v_0^{n-k}\ v_1^{k+s_\sigma}$.

For every k, $0 \le k \le n-1$,

$$\frac{b_\sigma}{B'_\sigma \rho_\sigma \sqrt[n]{a_2^k}} \tau_{\sigma,k+r_\sigma} x_{k+r_\sigma}^\sigma = \frac{1}{\sqrt[n]{a_2^k}} x_k \omega^{c(k)+k}.$$

Therefore, $\frac{b_\sigma}{B'_\sigma} \frac{x_{k+r_\sigma}^\sigma}{x_k} \in \mu_n$.

We want to compare this element with $B_\sigma \in L$:

$$B_\sigma = z_1 z_1^{-\sigma} M_\sigma \Rightarrow B_\sigma M_\sigma^{-1} = M_\sigma^{-1} B_\sigma = z_1 z_1^{-\sigma} \Rightarrow B_\sigma z_1^{-\sigma} = M_\sigma z_1.$$

By comparing $B_\sigma z_1^\sigma = \sum_{k=0}^{n-1} x_k^\sigma \omega^{c(k)+k} u_0^{n-k}\ u_1^k$ with $M_\sigma z_1 = \rho_\sigma u_0^{-r_\sigma}\ u_1^{r_\sigma}\ z_1$ we obtain $B_\sigma \frac{x_{k+r_\sigma}^\sigma}{x_k} \in \mu_n$. So $\frac{B_\sigma B'_\sigma}{b_\sigma} \in \mu_n$.□

The general expression of the elements γ_1 and γ_2 for $n = 3$ is

$$\begin{aligned}\gamma_1 = 3\ (&a^8 + 3(\omega+2)\ a^{22/3} + +9(\omega+2)\ a^{20/3}\ + 54\ a^6 + 216\ a^{16/3} \\ &+54(10-\omega)\ a^{14/3} + 1215\ a^4 + 162(10-\omega)\ a^{10/3} + 1944\ a^{8/3} + 1458\ a^2 \\ &+729(\omega+2)\ a^{4/3} + 729(\omega+2)\ a^{2/3} + 729),\end{aligned}$$

$$\begin{aligned}\gamma_2 = -3\ (&\omega^2\ y_2^6 + a_1\ (a_2\ y_1\ (y_1^2 - 3\ y_0\ y_2)\ \omega - \omega + (\omega+2)\ \sqrt[3]{a_2}\ y_1 \\ &+(\omega+2)\ a_2^{2/3}\ (y_1^2 - y_0\ y_2))\ y_2^3 + a_1^2\ (a_2\ y_1^3 - a_2^{5/3}\ (y_1^2 - y_0\ y_2)\ ((\omega-1)\ y_1^2 \\ &-3\ \omega\ y_0\ y_2)\ y_1 + (\omega+2)\ \sqrt[3]{a_2}\ y_1 + (\omega+2)\ a_2^{2/3}\ (y_1^2 - y_0\ y_2) \\ &+a_2^{4/3}\ ((1-\omega)\ y_1^4 + 3\ \omega\ y_0\ y_2\ y_1^2) + a_2^2\ (y_1^6 - 3\ y_0\ y_2\ y_1^4 + 3\ y_0^2\ y_2^2\ y_1^2) + 1))\end{aligned}$$

where $y_0 + y_1 \sqrt[3]{\omega a^{-1}} + y_2 \sqrt[3]{(\omega a^{-1})^2} = y$ is an element of $K(\sqrt[3]{\omega a^{-1}})$ such that $N_{K(\sqrt[3]{\omega a^{-1}})/K}(y) = \omega a_1^{-1} a_2^{-2}$.

For $n = 4$ the general expression can be found in Vela (1998a). The values of γ_1, γ_2 for $n = 3, 4$ and some specific values of a_1, a_2 are given in Tables 3 and 4. Tables for $n = 5, 6$ can also be found in Vela (1998a).□

12. Example 3: $0 \to \mathbb{Z}/n\mathbb{Z} \to E_2 \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to 0$

In Theorem 7.3 we study the embedding problem given by $L = K(\sqrt[n]{a_1}, \sqrt[n]{a_2})/K$ with $\Gamma = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ generated by h_1, h_2 such that $h_i(\sqrt[n]{a_j}) = \omega^{\delta_{ij}} \sqrt[n]{a_j}$ and the exact sequence

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} E_2 \xrightarrow{h} \Gamma \to 0$$

Table 3. Example 2. Results for $n = 3$.

a_1, a_2	γ_1, γ_2
$a_1 = \omega/4$	$\gamma_1 = 6\ (14835\ \sqrt[3]{2}\ (-\omega)^{2/3} + 10801\ 2^{2/3}\ \sqrt[3]{-\omega}\ + 27900)\ \omega$
	$-35874\ \sqrt[3]{2}\ (-\omega)^{2/3} + 43203\ 2^{2/3}\ \sqrt[3]{-\omega}\ + 73873$
$a_2 = -2$	$\gamma_2 = (-10 - 2\ \sqrt[3]{-2}\ + 7\ (-2)^{2/3})\ \omega + 7\ (-2)^{2/3} + 8\ \sqrt[3]{-2}\ - 2$
$a_1 = \frac{-(2+\omega)}{12}$	$\gamma_1 = 2(-4346757\sqrt[3]{6}\ (-\omega - 2)^{2/3} + 1388013\ 6^{2/3}\sqrt[3]{-\omega - 2}\ - 10164851)\omega$
	$-8106426\ \sqrt[3]{6}\ (-\omega - 2)^{2/3} + 1280445\ 6^{2/3}\ \sqrt[3]{-\omega - 2}\ - 32576255$
$a_2 = 2$	$\gamma_2 = 2^{2/3}\ \omega + 2^{2/3} - 2\sqrt[3]{6}\ + 2$
$a_1 = -1/4$	$\gamma_1 = 2\ (151470\ (-4\ \omega - 2)^{2/3} + 115197\ \sqrt[3]{-4\omega - 2} + 402913)\ \omega$
	$-69120\ (-4\ \omega - 2)^{2/3} + 352500\ \sqrt[3]{-4\omega - 2} + 402913$
$a_2 = 2\omega + 1$	$\gamma_2 = \omega\ (2\ (2\ \omega + 1)^2/3 + 3\ \sqrt[3]{2\omega + 1} + 3) + 5\ (2\ \omega + 1)^{2/3} + 3$
$a_1 = \omega^2/6$	$\gamma_1 = (1131408\ \sqrt[3]{6}\ (1 - \omega)^{2/3} + 1492989\ 6^2/3\ \sqrt[3]{1 - \omega} + 20329702)\ \omega$
	$-6971130\ \sqrt[3]{6}(1 - \omega)^{2/3} - 1000509\ 6^{2/3}\ \sqrt[3]{1 - \omega} - 12246553$
$a_2 = 2\ \omega + 1$	$\gamma_2 = -3\ \omega\ ((2\ \omega + 1)^{2/3} + 5\ \sqrt[3]{2\omega + 1} + 2) - 2\ (2\ \omega + 1)^{2/3}$
	$+9\ \sqrt[3]{2\omega + 1} + 15$
$a_1 = -\omega/8$	$\gamma_1 = 6\ \omega\ (1665629088\ ((\omega - 2)\ \omega)^{2/3} + 1239285406\ \sqrt[3]{(\omega - 2)\omega}$
	$+5040695061) - 2959464528\ ((\omega - 2)\ \omega)^{2/3}$
	$+8391883980\ \sqrt[3]{(\omega - 2)\omega} + 12306854125$
$a_2 = 2 - \omega$	$\gamma_2 = ()3 - 5\ \omega)\ ((4\ \sqrt[3]{2 - \omega} + 5)\ \omega + 4\ (2 - \omega)^{2/3} - \sqrt[3]{2 - \omega} + 4$

Table 4. Example 2. Results for $n = 4$.

a_1, a_2	γ
$a_1 = 2, a_2 = \omega/4$	$(1 + i)\ \sqrt[4]{i}\ \sqrt{2} + (-1)^{3/8}\ \sqrt{2}$
$a_1 = \omega, a_2 = -2$	$-4\ \sqrt[4]{-2} + 3\ \sqrt[4]{-1}2^{3/4}$
$a_1 = \omega, a_2 = 3$	$3^{3/4}$
$a_1 = -\omega, a_2 = -3$	$-6 - 3\imath - 3\ (1 + i)\ \sqrt[4]{-3} + 3\ \sqrt{3} + (1 + 3\imath)\ \sqrt[4]{-1}3^{3/4}$
$a_1 = 2, a_2 = -\omega - 1/2$	$(1 + 2\imath)\ \sqrt[4]{-\frac{1}{2} - i} + (1 - 2\imath)\ (-1/2 - i)^{3/4}$
$a_1 = 3, a_2 = 1 - \omega/3$	$28 - 16i - (11 + 33i)\sqrt[4]{1 - \frac{i}{3}} - (36 - 12i)\sqrt{1 - \frac{i}{3}} + (12 + 33i)(1 - i/3)^{3/4}$
$a_1 = 1/2, a_2 = \omega + 1$	$(1 + i)^{3/4}$

where E_2 is the group

$$E_2 = \langle \sigma, \tau\rho \mid \sigma, \tau, \rho \text{ have order } n, , \sigma\tau\sigma^{-1}\tau^{-1} = \rho, \sigma\rho = \rho\sigma, \tau\rho = \rho\tau \rangle.$$

If $n = p$ a prime integer, it is the *Heisenberg group of degree p*.

The obstruction to the solvability is the symbol $(\omega^{-1}a_1, a_2)$. We compute this obstruction with a representation which does not have degree 0. To apply Theorem 8.2 we need a representation of degree 0 such that $\varepsilon = s_t$.

We consider the representation: $t : \Gamma \to O(C), h_1 \mapsto \varphi(e_0^{-1}e_1), h_2 \mapsto \varphi(e_0^{-1}e_2)$, where $C = C_t = C(e_0, e_1, e_2) = C((-1)^{n-1}, 1, 1)$ with $A_t = \{x = \lambda e_0^{\epsilon_0} e_1^{\epsilon_1} e_2^{\epsilon_2}\} \subset F(C_t)$ and the admissible norm $\mathcal{N}(e_i) = 1 (i = 0, 1, 2)$. The twisted algebra by t is $\mathfrak{C} = C(v_0, v_1, v_2) = C((-1)^{n-1}a_1a_2, a_1a_2^2, a_2)$ where $v_0 = e_0 \otimes \sqrt[n]{a_1a_2}$, $v_1 = e_1 \otimes \sqrt[n]{a_1a_2^2}$ and $v_2 = e_2 \otimes \sqrt[n]{a_2}$ are fixed by the action of Γ. Clearly this representation is of degree 0 and it is easy to check that $\varepsilon = s_t$.

We suppose the problem is solvable. We want to find an element $\gamma \in L^*$ such that $M = L(\sqrt[n]{\gamma})$ is a solution. To simplify the notation we put $a = a_1a_2$.

The isomorphism g over L has the same expression as the one of Example 2 and we can express it as composition of two isomorphisms g_1 and g_2

$$C_L = C((-1)^{n-1}, 1, 1)_L \overset{g_1}{\simeq} C'_L = C((-1)^{n-1}a, a, 1)_L \overset{g_2}{\simeq} C((-1)^{n-1}a, aa_2, a_2)_L = \mathfrak{C}_L$$

given by $g_1(e_0) = \frac{1}{\sqrt[n]{a}}u_0$, $g_1(e_1) = \frac{1}{\sqrt[n]{a}}u_1$, $g_1(e_2) = u_2$, where the elements u_0, u_1 and u_2 generate $C((-1)^{n-1}a, a, 1)$ and $g_2(u_0) = v_0$, $g_2(u_1) = \frac{1}{\sqrt[n]{a_2}}v_1$, $g_2(u_2) = \frac{1}{\sqrt[n]{a_2}}v_2$. Clearly $g = g_2 \circ g_1$.

As before, we express the isomorphism f over K as composition of two simpler isomorphisms between generalized Clifford algebras generated by two elements

$$C = C((-1)^{n-1}, 1, 1) \overset{f_1}{\simeq} C'_L = C((-1)^{n-1}a, a, 1) \overset{f_2}{\simeq} C((-1)^{n-1}a, aa_2, a_2) = \mathfrak{C}$$

where the first one is obtained from the isomorphism $C((-1)^{n-1}, 1) \simeq C((-1)^{n-1}a, a)$ as in the above example and the second one from the isomorphism $C(a_1a_2, 1) \simeq C(a_1a_2^2, a_2)$ whose existence is given by the solvability of the embedding problem corresponding to a_1, a_2. We study the isomorphism $f_2 : C(a_1a_2, 1) \simeq C(a_1a_2^2, a_2)$ and working as in the above example we get the isomorphism

$$f_2 : C((-1)^{n-1}a, a, 1) \to C((-1)^{n-1}a, aa_2, a_2)$$

$$u_0 \mapsto v_0, \quad u_1 \mapsto \frac{1}{aa_2}\left(\sum_{k=0}^{n-2} y_{k+1}\ \omega^{c(k)} v_1^{n+1-k} v_2^k + !y_0 D^{-1} \omega^{c(n-1)} v_1^2 v_2^{n-1}\right),$$

$$u_2 \mapsto \frac{1}{aa_2}\sum_{k=0}^{n-1} y_k\ \omega^{c(k)} v_1^{n+1-k} v_2^k,$$

where $D = (-1)^{n-1}a^{-1}$ and $y_0 + y_1\sqrt[n]{D} + \cdots + y_{n-1}\sqrt[n]{D^{n-1}}$ is an element with norm $a_1^{-1}a_2^{-2}$.

We can modify $f(e_i)$ multiplying it by an adequate nth root of the unity if it is necessary, to assure $z \neq 0$.

As before, we compute the element z from the elements $z_1 \in C'_L$ and $z_2 \in \mathfrak{C}_L$ such that $g_i(x) = z_i f_i(x) z_i^{-1}$ for $i = 1, 2$, and x any element of the corresponding Clifford algebra and the element $z = g_2(z_1)\, z_2 \in \mathfrak{C}_L$ satisfies $g(x) = zf(x)z^{-1}$ for all $x \in C_L$. Since the isomorphisms f_1 and g_1 have the same expression as the ones of the above example, the element z_1 has the same coordinates in the corresponding basis. We compute the element z_2 as before, from f_2 and g_2.

We now compute the element $\gamma \in L$ such that $M = L(\sqrt[n]{\gamma})$ is a solution to the embedding problem. Working as before, the element $\gamma = \gamma_1 \cdot \gamma_2$ where

$$\gamma_1 = \eta_1\alpha_1^{-1},\ \alpha_1 \text{ is a coordinate of } N(z_1) \text{ and } \eta_1 = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \sqrt[n]{(a_1a_2)^{n/2}} & \text{if } n \text{ is even} \end{cases}$$

and

$$\gamma_2 = \eta_2\alpha_2^{-1},\ \alpha_2 \text{ is a coordinate of } N(z_2) \text{ and } \eta_2 = 1.$$

The general expression of the elements γ_1 and γ_2 for $n = 3$ is

$$\gamma_1 = 3\ (a^8 + 3(\omega+2)\ a^{22/3} + 9(\omega+2)\ a^{20/3}\ + 54\ a^6 + 216\ a^{16/3}$$

$$
\begin{aligned}
&+54(10-\omega)\ a^{14/3}+1215\ a^{4}+162(10-\omega)\ a^{10/3}+1944\ a^{8/3}\\
&+1458\ a^{2}+729(\omega+2)\ a^{4/3}+729(\omega+2)\ a^{2/3}+729)\\
\gamma_2 = {}& y_2^6 - a_1\ y_2^3\ (1+(1+2\ \omega)\ \sqrt[3]{a_2}\ y_1+(1+2\ \omega)\ a_2^{2/3}\ (y_1^2-y_0\ y_2)\\
&-a_2\ (y_1^3-3\ y_0\ y_1\ y_2))+a_1^2\ (1+(2+\omega)\ \sqrt[3]{a_2}\ y_1+a_2\ y_1^3\\
&+(2+\omega)\ a_2^{2/3}\ (y_1^2-y_0\ y_2)-a_2^{5/3}\ y_1\ (y_1^2-y_0\ y_2)\ (-y_1^2+\omega\ y_1^2-3\ \omega\ y_0\ y_2)\\
&+a_2^{4/3}\ (y_1^4-\omega\ y_1^4+3\ \omega\ y_0\ y_1^2\ y_2)+a_2^2\ (y_1^6-3\ y_0\ y_1^4\ y_2+3\ y_0^2\ y_1^2\ y_2^2))
\end{aligned}
$$

where $y_0 + y_1\sqrt[3]{a^{-1}} + y_2\sqrt[3]{(a^{-1})^2} = y \in K(\sqrt[3]{a^{-1}})$ has $N_{K(\sqrt[3]{a^{-1}})/K}(y) = a_1^{-1}a_2^{-2}$.

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